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THE SCHWARZ-PICK LEMMA FOR CIRCLE PACKINGS

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Connections between circle packings and analytic functions were suggested by William Thurston at the International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University, March 1985. He conjectured that the conformal mapping of a simply connected plane domain Ω to the unit disc Δ could be approximated by manipulating hexagonal circle configurations lying in Ω. His idea is illustrated in Figure 1: First, approximate Ω with a uniform hexagonal circle packing P as in 1(a). Now, repack P to obtain a certain combinatorially equivalent extremal circle packing P_a lying in Δ, as shown in 1(b). Finally, define a piecewise affine mapping from P_a to P by identifying centers of corresponding circles in the two configurations. He conjectured that as the sizes of the circles in P go to zero (and assuming certain natural normalizations), the mappings so defined would converge uniformly on compact subsets of Δ to the conformal (analytic) mapping of Δ onto Ω. This conjecture was subsequently proven by Burt Rodin and Dennis Sullivan [6].

Thurston termed the conjectured result a "Finite Riemann Mapping Theorem", the intuition being, at least in part, that since the conformal map carries infinitesimal circles to infinitesimal circles, one might approximate it by mapping real circles to real circles. Our interest is in developing this analogy—rather than consider increasingly fine packings and the approximation question, we will study individual circle packings P and how they compare to their extremal repackings P_a. Our main result is a natural analogue for the classical Schwarz Lemma, in the invariant form due to Pick, which we term the "Discrete Schwarz-Pick Lemma" or DSPL. In the definitions and results along the way, further parallels with classical complex analysis emerge. There are several intriguing results on circle packings in Chapter 13 of Thurston's notes [10], and certain of the key ideas here (e.g., parameterized hyperbolic structures and some monotonicity results) occur there, though without our complex function theory slant.

We want to state our main result somewhat informally here at the beginning before introducing more technical definitions and notation. To that

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