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INVERSE REAL CLOSED SPACES

BY

NIELS SCHWARTZ

Introduction

Building up semi-algebraic geometry, H. Delfs and M. Knebusch relied almost entirely on geometric arguments (cf. [5]). To develop algebraic methods appropriate for the purposes of semi-algebraic geometry the rings of sections of the structure sheaves of semi-algebraic spaces must be studied. First steps in this direction were taken in [13], [14], [15], [16], [17] (see also [2]). An attempt to develop an algebraic version of semi-algebraic geometry with a sufficient degree of generality and at the same time keeping the connections with geometry leads to a category of locally ringed spaces, called real closed spaces [13], [15], [16]. This class of spaces generalizes locally semi-algebraic spaces much in the same way as schemes generalize classical algebraic varieties.

Using weakly semi-algebraic spaces [10], M. Knebusch has been particularly successful developing algebraic topology for semi-algebraic spaces. These spaces are obtained from affine semi-algebraic spaces by glueing them together on closed semi-algebraic subspaces. From a purely algebraic point of view these spaces are nothing new since their rings of sections are also real closed rings [16, Chapter I, §4]. However, to keep close connections between algebra and geometry the development of an algebraic version of the weakly semi-algebraic spaces requires the construction of a new class of spaces: affine real closed spaces have to be glued together on closed constructible subspaces.

Recall that an affine real closed space is a pro-constructible subspace of the real spectrum of a ring together with a sheaf of real closed rings [15], [16]. To glue two such spaces together on closed constructible subspaces and still be able to use the usual sheaf theoretic techniques, the notion of openness is redefined: The *inverse topology* on the pro-constructible subset K of the real spectrum $\mathrm{Sper}(A)$ of the ring A has the closed (in the usual topology) constructible subsets of K as its basis. If K with the inverse topology is denoted by K^*, then K^* can be equipped with a structure sheaf of real

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