ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 3, Fall 1992

ON SEMIMARTINGALE DECOMPOSITIONS OF CONVEX FUNCTIONS OF SEMIMARTINGALES

BY

ERIC CARLEN[1] AND PHILIP PROTTER[2]

Let X be a semimartingale with values in $\mathbf{R}^d$, and let $X_t = X_0 + M_t + A_t$ be a decomposition of X into a local martingale M and a càdlàg, adapted, finite variation process A, with $M_0 = A_0 = 0$. Let $f: \mathbf{R}^d \to \mathbf{R}$ be convex. P.A. Meyer showed in 1976 [6] that $f(X)$ is again a semimartingale. We will give a new proof of this result which moreover gives the semimartingale decomposition of $f(X)$ in terms of uniform limits of explicitly identified processes.

The case where $d = 1$ is already well understood. Indeed, the Meyer-Tanaka formula allows us to give an explicit decomposition of $f(X)$:

$$(1)\qquad f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\, dM_s + \left\{ \int_0^t f'(X_{s-})\, dA_s + \frac{1}{2}\int_{\mathbf{R}} L_t^a \mu(da) + \sum_{0<s\le t} \left(f(X_s) - f(X_{s-}) - f'(X_s)\,\Delta X_s\right)\right\},$$

where f' is the left continuous version of the derivative of f, L_t^a is the local time of X at the level a, the measure μ is the second derivative of f in the generalized function sense, and the term in brackets $\{\cdots\}$ is the finite variation term in a decomposition of $f(X)$. See [8] for details on this formula. Moreover in the case $d = 1$ if B is a standard Brownian motion and $f(B)$ is a semimartingale, then f must be the difference of two convex functions (see [3]), hence convex functions are the most general functions taking semimartingales into semimartingales.

We now turn to the case $d \ge 2$, where $f: \mathbf{R}^d \to \mathbf{R}$ is convex. Except in very special cases (see [2], [4], [5], [7], [9], [10]) no formula such as (1) is known to exist, except of course when f is $\mathscr{C}^2$, and then the Meyer-Itô formula gives

---

Received October 23, 1990.
1991 Mathematics Subject Classification. Primary 60H05; Secondary 60G44, 60G48.
[1]NSF Postdoctoral Fellow.
[2]Supported in part by a grant from the National Science Foundation.