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A NEW CHARACTERIZATION OF DIRICHLET TYPE SPACES AND APPLICATIONS

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1. Introduction

Let $\mathbf{D}$ be the unit disk of the complex plane $\mathbf{C}$ and $dA(z) = 1/\pi\, dx\, dy$ be the normalized Lebesgue measure on $\mathbf{D}$. For $\alpha < 1$, let

$$dA_\alpha(z) = (2 - 2\alpha)(1 - |z|^2)^{1-2\alpha}\, dA(z).$$

The Sobolev space $L^{2,\alpha}$ is the Hilbert space of functions $u\colon \mathbf{D} \to \mathbf{C}$, for which the norm

$$\|u\| = \left(\left|\int_{\mathbf{D}} u\, dA_\alpha(z)\right|^2 + \int_\Delta \left(|\partial u/\partial z|^2 + |\partial u/\partial \bar{z}|^2\right) dA_\alpha(z)\right)^{1/2}$$

is finite. The space D_α is the subspace of all analytic functions in $L^{2,\alpha}$. This scale of spaces includes the Dirichlet type spaces ($\alpha > 0$), the Hardy space ($\alpha = 0$) and the Bergman spaces ($\alpha < 0$). (The Hardy and Bergman spaces are usually described differently, however see Lemma 3 of Section 3.) Let

$$\dot{D}_\alpha = \{g \in D_\alpha : g(0) = 0\}$$

and let

$$\dot{P} = \{g \text{ is a polynomial on } \mathbf{D} : g(0) = 0\}.$$

Clearly $\dot{P}$ is dense in $\dot{D}_\alpha$. Let P_α denote the orthogonal projection from $L^{2,\alpha}$ onto $\dot{D}_\alpha$. For a function $f \in L^{2,\alpha}$ it is possible to define the (small) Hankel operator with symbol f, $h_f^{(\alpha)}$, on $\dot{P}$ by (see also [W1])

$$h_f^{(\alpha)} = \overline{P_\alpha(f\bar{g})}.$$

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