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RANDOM ELEMENTS OF A FREE PROFINITE GROUP GENERATE A FREE SUBGROUP

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Consider each profinite group as a probability space, the probability being the normalized Haar measure. Jarden proved that almost all $z \in \hat{\mathbf{Z}}$ generate a closed subgroup of infinite index while almost all k-tuples with $k \geq 2$ generate an open subgroup [FJ, Lemma 16.15]. Moreover, the closed subgroup of $\hat{\mathbf{Z}}$ generated by an e-tuple $(z_1, \ldots, z_e)$ which is chosen at random is isomorphic to $\hat{\mathbf{Z}}$. Fried and Jarden ask for $e \geq 2$ about the probability that a e-tuple $(x_1, \ldots, x_e) \in \hat{F}_e$ generates a closed subgroup which is isomorphic to $\hat{F}_e$ and about the probability that a e-tuple of elements of $\hat{F}_e$ generates an open subgroup [FJ, Problem 16.16]. Here $\hat{F}_e$ is the free profinite group of rank e.

W. M. Kantor and the present author show [KL] that the second probability is 0. The aim of this note is to prove that the first probability is 1. Actually the full result is somewhat more general:

THEOREM 1. *Let F be a free profinite group of rank at least 2, and let k be a positive integer.*

(a) *The probability that a k-tuple of elements of F generates an open subgroup is* 0.

(b) *The probability that a k-tuple of elements of F generates a closed subgroup which is isomorphic to $\hat{F}_k$ is* 1.

As mentioned, part (a) is proved in [KL]. We supply a proof which replaces the use of Dixon's theorem by more elementary arguments. Some of the ingredients of the proof of (a) are also used in the proof of (b).

Notation. For a finite group and a positive integer e let

$$d_e(G) = \max\{m \in \mathbf{N} | G^m \textit{ is generated by } e \textit{ elements}\}$$

$$D_e(G) = \{(x_1, \ldots, x_e) \in G^e | \langle x_1, \ldots, x_e \rangle = G\}$$

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