ILLINOIS JOURNAL OF MATHEMATICS
Volume 37, Number 3, Fall 1993

THE GROWTH OF $\Phi\Phi''/\Phi'^2$ FOR CONVEX FUNCTIONS

P.C. FENTON

1. Introduction

In W.K. Hayman's survey of the Wiman-Valiron method, the following growth lemma is proved:

LEMMA A [2, Lemma 9]. *Let $\Phi(r)$ be a positive, increasing and convex function of r for $r \geq r_0$ and suppose that*

$$(1.1) \qquad \liminf_{r\to\infty} \frac{\log \Phi(r)}{\log r} \leq \rho \leq \limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r},$$

where $\rho > 1$. Let $\alpha(\rho) = (\rho - 1)/\rho$ if $\rho < \infty$; $\alpha(\rho) = 1$ if $\rho = \infty$. Suppose that a, K are constants such that $K > 1$ and $a < \alpha(\rho)$. Then if E is the set of all r such that

$$(1.2) \qquad \textit{either (a)} \ \frac{\Phi(r)\Phi''(r)}{\Phi'(r)^2} > K\alpha(\rho) \ \textit{or (b)} \ \Phi'(r) \leq \Phi(r)^a,$$

we have $\underline{\text{dens}}\, E \leq K^{-1}$, where "$\underline{\text{dens}}$" is the lower (linear) density.

Hayman applies this in a context the details of which need not detain us here, save to say that his results suggest that, when ρ is the upper limit in (1.1), it would be desirable to strengthen the part of the conclusion of the lemma that concerns E, from lower to upper density. This is evidently not possible, however, since Φ may be linear for arbitrarily long stretches, and (1.2b) itself may therefore hold on a set of upper density 1. In Hayman's argument it is (1.2a) that plays the vital role, (1.2b) being subsidiary in the sense that it is used only to show that an error term is inessential. What can be said about the set on which (1.2a) holds? It is entirely with this question that the remainder of the present note is concerned.

Received September 9, 1991.
1991 Mathematics Subject Classification. Primary 26A51.