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ON A THEOREM OF AKHIEZER

ELIZABETH KOCHNEFF, YORAM SAGHER AND RUBY TAN

1. Introduction

Akhiezer, [1], showed for $f \in L^2(R)$, $\gamma = a + ib$, $b \neq 0$, that

$$(t-\gamma)H\left(\frac{f(x)}{x-\gamma}\right)(t) = Hf(t) - C(\gamma, f)$$

where H is the Hilbert transform and $C(\gamma, f)$ is a constant depending on f and γ.

If γ is real and both $f \in L^2(R)$ and $(f(t) - \alpha)/(t - \gamma) \in L^2(R)$, Akhiezer showed

$$(t-\gamma)H\left(\frac{f(x)-\alpha}{x-\gamma}\right)(t) = Hf(t) - C(\gamma, f).$$

Akhiezer's proof depends on calculations of Fourier transforms, using complex methods, and therefore does not seem to generalize to $p \neq 2$. A much simpler proof of Akhiezer's theorem in the case $\alpha = 0$ is given in [3]. We prove the theorem under the hypotheses

$$f \in L^1\big(R, dt/(1+t^2)\big) \quad \text{and} \quad (f(t)-\alpha)/(t-\gamma) \in L^1_{loc}.$$

For $\gamma \in R$, if $f \in L^1(R, dt/(1 + |t|))$ or if $f \in L^1(T)$, and if $(f(t) - \alpha)/(t - \gamma) \in L^1_{loc}$, we show that $Hf(\gamma)$ exists and equals $C(\gamma, f)$. Since we may

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