ILLINOIS JOURNAL OF MATHEMATICS
Volume 37, Number 3, Fall 1993

ON A GENERALIZED ARTIN-SCHREIER THEOREM FOR REAL-MAXIMAL FIELDS

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Introduction

A celebrated theorem of Artin and Schreier [AS] characterizes the real closed fields as the fields K whose absolute Galois group $G(K)$ consists of precisely two elements. A natural generalization of the class of real closed fields is the class of *real-maximal* fields, i.e., fields K which have no proper separable algebraic extension to which all the orderings of K extend. Thus, a real-maximal field with no orderings is nothing but a separably closed field, and a real-maximal field with precisely one ordering is just a real closed field. We prove:

THEOREM A. *Let $0 \leq e \leq 3$. The following conditions on a field K are equivalent*:
(a) *K is real-maximal with precisely e orderings*;
(b) *$G(K)$ is isomorphic to the free pro-2 product D_e of e copies of $\mathbf{Z}/2\mathbf{Z}$.*

For $e = 1$ this is the Artin-Schreier theorem, while for $e = 2$ it has been proved by Bredikhin, Eršov and Kal'nei using other methods [BEK]. For $e \geq 4$, however, this equivalence is no longer true: Although (b) still implies (a), one can construct real-maximal fields with e orderings whose absolute Galois group is not D_e (Example 2.7). Nevertheless, one has the following result due to Kal'nei, mentioned without proof in [E1] and generalized in Corollary 1.5 below:

THEOREM B. *Let K be a real-maximal field with precisely e orderings $P_1, \ldots, P_e$ and assume that $P_1, \ldots, P_e$ induce distinct order topologies on K. Then $G(K) \cong D_e$.*

As was shown by van den Dries [D, Ch. II] and in subsequent works by Prestel [P1] and Jarden [J1], the real-maximal fields also arise naturally in

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Received August 21, 1991.
1991 Mathematics Subject Classification. Primary 12D15; Secondary 12J15.
[1]Partially supported by grants from Lion's Foundation and the G.I.F., the German-Israeli Foundation for Scientific Research and Development.