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DEFORMATIONS AND DIFFEOMORPHISM TYPES OF HOPF MANIFOLDS

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1. Introduction

A generalized Hopf manifold or simply a Hopf manifold of complex dimension n is a compact complex manifold of which the universal covering is $\mathbf{C}^n - \{\mathbf{0}\}$, where n is a positive integer ($n \geq 2$).

The Hopf manifold, first introduced by H. Hopf, is well known as the first example of a non-Kähler manifold. In his essays [3] presented to R. Courant, H. Hopf referred to a complex manifold diffeomorphic to $S^1 \times S^{2n-1}$, which was originally called a Hopf manifold. The generalized definition above is due to K. Kodaira [6].

Perhaps one of the first fundamental problems concerning the Hopf manifold is to determine their diffeomorphism types. This was done for the case of $n = 2$ by M. Kato [4]. Later, in his paper [5], M. Kato studied submanifolds of Hopf manifolds and obtained a result on diffeomorphism types of Hopf manifolds (although the result is not fully stated as a theorem, it may be inferred from the results in the paper).

In this paper we study deformations of Hopf manifolds and give a short and direct proof of the theorem that a Hopf manifold of complex dimension n is diffeomorphic to a fiber bundle over S^1 with fiber S^{2n-1}/H, defined by a representation $\rho\colon \pi_1(S^1) \to N_{U(n)}(H)$ such that $\rho(1)$ is an element of finite order in $N_{U(n)}(H)$, where H is a finite unitary and fixed-pint-free group, and $N_{U(N)}$ is the nomalizer of H in $U(n)$. This theorem determines explicitly the diffeomorphism types of the Hopf manifolds.

We state here a conjecture that a compact complex manifold of which the universal covering is $\mathbf{C}^n$ is diffeomorphic to a manifold which has a torus or a non-toral nilmanifold as a finite covering. The first case is clearly a Kähler manifold and the second case is a non-Kähler manifold (cf. [2]).

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