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CONVERGENCE OF ERGODIC AVERAGES ON LATTICE RANDOM WALKS

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1. Introduction

This note is concerned with a question posed by H. Furstenberg and communicated to the author by Y. Katznelson.[2] For $d \geq 1$, consider the standard random walk on the lattice $\mathbf{Z}^d$, i.e., if $n(j) \in \mathbf{Z}^d$ in the j^{th} position, then $n(j+1)$ takes one of the values

$$n(j) \pm e_1, n(j) \pm e_2, \dots, n(j) \pm e_d$$

with equal probability. Here e_j stands for the j^{th} unit vector. We consider ergodic averages along the sequence $\{n(j)\}$. Thus take some probability space (Ω, μ) and d commuting, invertible measure preserving transformations $T_1, T_2, \dots, T_d$ of Ω. Given a measurable function f on Ω, define

$$A_k f = \frac{1}{k} \sum_{j=1}^{k} T_1^{n(j,1)} T_2^{n(j,2)} \dots T_d^{n(j,d)} f \tag{1.1}$$

where $n(j) = (n(j,1), \dots, n(j,d))$. We are interested in the convergence properties of the averages (1.1). In this respect, we will prove the following

THEOREM. *Almost any random walk $\{n(j)\}$ on $\mathbf{Z}^d$ has the property that given any system $(\Omega, \mu, T_1, \dots, T_d)$ of commuting transformations and $f \in L^p(\Omega, \mu)$, $p > 1$, the averages $A_k f$ given by* (1.1) *converge almost surely, along any sequence $\{k_s\}$ satisfying $k_{s+1} > k_s \log\log k_s$. In particular, there is convergence for the logarithmic averages. If moreover, one of the transformations is ergodic, the limit is $\int_\Omega f\,d\mu$.*

A comment on the restrictions made in the statement: The subsequence extraction is needed, even for $d = 1$, and a generic random walk $\{n(j)\}$ is not

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