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CONVEXITY OF THE GEODESIC DISTANCE ON SPACES OF POSITIVE OPERATORS

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Let A be a C^*-algebra with 1 and denote by A^+ the set of positive invertible elements of A. The set A^+ being open in $A^s = \{a \in A;\ a^* = a\}$ it has a C^∞ structure and we can identify TA_a^+ with A^s for each $a \in A^+$. We use G to denote the group of invertible elements of A. Notice that G operates on the left on A^+ by the rule

$$L_g a = (g^*)^{-1} a g^{-1} \quad (g \in G, a \in A^+).$$

This action allows us to introduce a natural reductive homogeneous space structure in the sense of [8] (for details see [2], [3], [4]).

The corresponding connection—which is preserved by the group action—has covariant derivative

$$\frac{DX}{dt} = \frac{dX}{dt} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma})$$

where X is a tangent field on A^+ along the curve γ and exponential

$$\exp_a X = e^{Xa^{-1}/2} a e^{a^{-1}X/2}, \quad a \in A^+, X \in TA_a^+.$$

The curvature tensor has the formula

$$R(X,Y)Z = -\tfrac{1}{4}a\big[[a^{-1}X, a^{-1}Y], a^{-1}Z\big]$$

for $X, Y, Z \in TA_a^+$. The manifold A^+ has also a natural Finsler structure given by

$$\|X\|_a = \|a^{-1/2}Xa^{-1/2}\| \text{ for } X \in TA_a^+$$

and the group G operates by isometries for this Finsler metric.

THEOREM 1. *If $J(t)$ is a Jacobi field along the geodesic $\gamma(t)$ in A^+ then $\|J(t)\|_{\gamma(t)}$ is a convex function of $t \in \mathbf{R}$.*

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