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RESTRICTIONS OF FOURIER TRANSFORMS TO FLAT CURVES IN $\mathbf{R}^2$

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1. Introduction

Given a smooth (lower-dimensional) submanifold S of $\mathbf{R}^n$ and a smooth compactly supported measure σ on S, one may ask for what values of p and q an *a priori* estimate of the form

$$\|\hat{f}|_S\|_{L^q(\sigma)} \leq C_{p,q}\|f\|_{L^p(\mathbf{R}^n)} \quad \forall f \in \mathscr{S}(\mathbf{R}^n) \tag{1.1}$$

holds, where $\hat{f}|_S$ denotes the restriction of the Fourier transform of f to S, and $\mathscr{S}(\mathbf{R}^n)$ is the Schwartz class of functions. Estimates of this type are known as *restriction* theorems. Note that if $p = 1$ the estimate holds trivially (for any q). On the other hand, if $p = 2$ such an estimate cannot hold, since S has Lebesgue measure zero in $\mathbf{R}^n$. E. M. Stein was the first to observe that a restriction theorem holds for $q = 2$ and some $p > 1$ when S is the n-sphere, or more generally, when an estimate of the form

$$|\hat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-\varepsilon} \; \forall \xi \in \mathbf{R}^n \tag{1.2}$$

holds with some $\varepsilon > 0$ for the Fourier transform of the measure σ on S (see [F], [S]). The estimate (1.2) holds, for instance, if S is of *finite type*, namely each point of S has at most a finite order contact with any hyperplane. Hence it follows that (1.1) holds for all finite type S with $q = 2$ and a nontrivial p, that is, some $p \in (1, 2)$. See [S] for more details. Also see [F],[T], [Z], [C], [DM], [So] and further references cited in those works.

On the other hand, it is well known that a nontrivial restriction estimate need not hold if the curvature vanishes to *infinite* order at some point of S in such a way that (1.2) should fail—we will call such S (infinitely) *flat*. (So the surface of a circular cylinder, say, is not flat, since (1.2) holds for it.) For example, if S is the flat curve in $\mathbf{R}^2$ given as the graph of the function $\gamma(t) = e^{-1/t^2}$ near the origin, then a homogeneity argument shows that (1.1) fails for *every* $p > 1$. However, in this paper we show that an analog of (1.1) does hold for a class of strictly convex curves whose curvature vanishes, to infinite (or finite) order, at the origin, where the L^p space on the right side of

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