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THE HENSTOCK AND MCSHANE INTEGRALS OF VECTOR-VALUED FUNCTIONS

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Introduction

A familiar formula from undergraduate analysis is the 'Riemann sum' $\sum_{i=1}^{n} f(t_i)(b_i - b_{i-1})$ of a function f with respect to a tagged partition $0 = b_0 \leq t_1 \leq b_1 \leq \cdots \leq t_n \leq b_n = 1$ of $[0, 1]$. One of the standard definitions of the Riemann integral describes it as the limit of such sums as $\max_{1 \leq i \leq n}(b_i - b_{i-1}) \to 0$. It is a remarkable fact that the same formula may be used to define a vastly more powerful integral, if we take a different limiting process. Instead of requiring all partitions with $\max_i(b_i - b_{i-1}) \leq \delta_0$ to give good approximations to the integral, we can restrict our attention to those in which $b_i - b_{i-1} \leq \delta(t_i)$ for each i, where δ is a strictly positive function on $[0, 1]$. (See 1(c) below.) This refinement yields the 'Henstock' or 'Riemann-complete' integral; it agrees with the Lebesgue integral on non-negative functions but extends it on others (see 4(e) below). An ingenious modification of the construction, due to E.J. McShane, allows the t_i to lie outside the corresponding intervals (see 1(b)); this brings us back a step, to the Lebesgue integral precisely.

A common feature of the Riemann, McShane and Henstock integrals is that the use of Riemann sums gives us obvious formulations of integrals for vector-valued functions defined on $[0, 1]$. For the McShane and Henstock integrals I spell these out in 1(*b*-*c*) below. The Henstock integral obviously extends the McShane integral. In this paper I seek to elucidate the nature of this extension; in particular, to give criteria to distinguish McShane integrable functions among the Henstock integrable functions. In the real-valued case this is simple enough; a Lebesgue integrable function is just a Henstock integrable function with (Henstock) integrable absolute value; equivalently, a Henstock integrable function which is Henstock integrable over every measurable set. It turns out that the latter criterion is valid in the vector-valued case (Corollary 9 below). I give priority however to a more economically expressible result in terms of the Pettis integral: a vector-valued function is McShane integrable iff it is both Henstock integrable and Pettis integrable (Theorem 8). The Pettis integral being the widest of the standard integrals of vector-valued functions (see [7]), this suggests that the difference between the

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