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INHOMOGENEOUS INEQUALITIES OVER NUMBER FIELDS

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1. Introduction

In the classical theory of diophantine approximation, Kronecker, in 1884, was the first to investigate inhomogeneous approximation to real linear forms which were, in some sense, independent over **Z**. In a slightly different direction, Khintchine, in 1936, proved that if a homogeneous system of real linear forms was not approximated well by integers (i.e., it was a badly approximable system), then this implied the existence of an excellent integer approximation to any associated inhomogenous system (see, for example, [9]). Here we study related inhomogeneous problems in the setting of an arbitrary number field. In particular, we examine these issues in the context of the ring of S-integers and over the associated adèle ring of the number field. Diophantine approximation over the adèle ring was first studied by Cantor in 1965 [4], then later by Sweet [12] and more recently by the author [2].

In Section 2 we precisely describe all our notation, but briefly, let k be a number field and S a finite collection of places of k containing all archimedean places. We write $k_S = \prod_{v \in S} k_v$ for the topological product of the completions k_v. Let $\{A_v\}_{v \in S}$ be a collection of $M \times N$ matrices such that for each $v \in S$, A_v has its entries over k_v. The S-system $\{A_v\}_{v \in S}$ is said to be a *badly approximable S-system of linear forms* if there exists a constant $\tau > 0$ such that

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v \in S} |A_v \vec{x} - \vec{y}|_v^M$$

for all S-integer column vectors $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{x} \neq \vec{0}$ and $\vec{y} \in (\mathscr{O}_S)^M$, where h_S is a suitably normalized S-height. Our first result is a generalization of Khintchine's theorem to number fields.

THEOREM 1. *Let $\{A_v\}_{v \in S}$ be a badly approximable S-system of dimension $M \times N$. For each $v \in S$ suppose that $\varepsilon_v \in k_v$ satisfies $0 < \|\varepsilon_v\|_v < 1$. Then for any $\vec{\beta} = (\vec{\beta}_v) \in (k_S)^M$, there exist vectors $\vec{x} \in (\mathscr{O}_S)^N$ and $\vec{y} \in (\mathscr{O}_S)^M$ such*

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