ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 3, Fall 1994

FUNCTIONAL INEQUALITIES, JACOBI PRODUCTS, AND QUASICONFORMAL MAPS

M.K. VAMANAMURTHY AND M. VUORINEN

1. Introduction

The special function (cf. (2.1))

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K), \tag{1.1}$$

where $K \in (0,\infty)$, $r \in (0,1)$, is closely related to geometric properties of quasiconformal mappings. Some examples of such geometric properties are the quasiconformal Schwarz lemma [LV, p. 64] and the study of the Beurling-Ahlfors extension of quasisymmetric functions [AH], [L], [LV]. We first recall two earlier explicit estimates for the function $\varphi_K(r)$ and then give our main results, which yield new identities and inequalities for this frequently occurring function. The basic inequality

$$r^{1/K} < \varphi_K(r) < 4^{1-1/K} r^{1/K} \tag{1.2}$$

for $K \in (1,\infty)$ and $r \in (0,1)$, has been known for more than thirty years. This inequality was recently sharpened [AVV3] to

$$\frac{1}{\operatorname{ch}\left(\frac{1}{K}\operatorname{arch}\left(\frac{1}{r}\right)\right)} < \varphi_K(r) < \operatorname{th}(\operatorname{arth} r + (K-1)\mu(r')), \tag{1.3}$$

for $K \in (1,\infty)$, $r \in (0,1)$ with $r' = \sqrt{1-r^2}$.

1.4. THEOREM. *For $K \in (0,\infty)$, let $f\colon [0,1] \to R$, be defined by*

$$f(r) = \frac{1-\varphi_{1/K}(r)}{(1-r)^{1/K}} \quad \textit{for } 0 \le r < 1,$$

and $f(1) = 8^{1-1/K}$. Then f is strictly increasing if $K > 1$ and strictly decreasing

Received June 19, 1992.
1991 Mathematics Subject Classification. Primary 30C62; Secondary 33E05.