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SOME SINGULAR SERIES AVERAGES AND THE DISTRIBUTION OF GOLDBACH NUMBERS IN SHORT INTERVALS

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1. Introduction

The Goldbach conjecture asserts that every even integer exceeding two can be written as the sum of two primes. As this has still not been substantiated, there is reason to distinguish those even integers which can be written as the sum of two primes; we call such an integer a Goldbach number. There are many results that have been proven about Goldbach numbers (see, for example, the introduction in [Go_2]).

In this paper we shall be concerned with the question of the existence of Goldbach numbers in short intervals and the asymptotic formula for the number of representations of the even integers in a short interval as the sum of two primes.

It was proven by Montgomery and Vaughan [MV_2] that every interval $(N-K, N]$ contains Goldbach numbers provided that $K > N^{7/72+\varepsilon}$ and $N > N_0(\varepsilon)$. More recently Perelli and Pintz [PP] have proven that almost every even integer in the interval $(N-K, N]$ is a Goldbach number if $K > N^{7/36+\varepsilon}$.

In the case that one admits conditional results it is possible to treat significantly shorter intervals and there has been a history of results based on certain unproved hypotheses. The first such result (which preceded the unconditional results) was due to Linnik [L_1] who proved, under the assumption of the Riemann Hypothesis, that one could find Goldbach numbers in every interval $(N-K, N]$ with $K > (\log N)^{3+\varepsilon}$ and $N > N_0(\varepsilon)$. Linnik's result was sharpened by Kátai [K] and later but independently by Montgomery and Vaughan [MV_2] so as to replace $(\log N)^{3+\varepsilon}$ by $C\log^2 N$ for a suitable absolute constant C, again under the assumption of the Riemann Hypothesis.

The next step was taken by Goldston [Go_2]. To describe this we need to define the integral

$$J(N,h) = \int_1^N (\psi(x+h) - \psi(x)D - h)^2\, dx, \tag{1.1}$$

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