ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 1, Spring 1995

THE GENERALIZED MCSHANE INTEGRAL

D.H. FREMLIN

Introduction

An interesting definition of the Lebesgue integral on $[0,1]$, as a limit of suitable Riemann sums, was developed by E.J. McShane in a series of papers; see [16] for a full account of this, including extensions to such spaces as $\mathbf{R}^n$ and $\mathbf{R}^{\mathbf{N}}$, as indicated in 1G below. It is characteristic of such definitions of the integral that they are readily adaptable to provide a theory of integration for vector-valued functions, and this was done for the McShane integral on $[0,1]$ by R.A. Gordon [13]. McShane was primarily concerned to provide an intuitively and technically straightforward construction of the Lebesgue interval, and made no attempt to push his method to the most general case. My aim in this paper is to show that, with a little effort, a successful generalization can be found, which can deal with functions from any of a wide variety of topological measure spaces to a Banach space, is related in interesting ways to other known integrals, and has a satisfying number of properties of its own.

The context in which I work is that of 'σ-finite outer regular quasi-Radon measure spaces' (see 1Ba–c below); this covers most of the important topological measure spaces which have been described. The paper has four sections.

1. I begin by defining the integral (1A–1B) and showing that it does indeed agree with Gordon's version when the domain space is $[0,1]$, and with McShane's versions when the range space is $\mathbf{R}$ and the domain space is one of those he considers (1C–1G). I continue by showing that the McShane integral lies between the Bochner and Pettis integrals (1K, 1Q), and in particular always agrees with the ordinary integral when the range space is $\mathbf{R}$ (1O).

2. In the second section I give some results of a technical type, showing that 'lim sup' in the definition of the integral may be replaced by a simple limit (2D) and that the two natural definitions of $\int_E \phi$ agree for measurable sets E(2E–2F).

3. I then describe the relationship between the McShane and Talagrand integrals; this follows the lines established in [10] for the case in which the domain space is $[0,1]$. If the unit ball of X^* is w^*-separable, then an

Received September 2, 1992.
1991 Mathematics Subject Classification. Primary 28B05; Secondary 46G10.