ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 3, Fall 1995

ALGORITHMS FOR THE COMPLETE DECOMPOSITION OF A CLOSED 3-MANIFOLD

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0. Introduction

Let F be a properly embedded normal surface in a compact, triangulated 3-manifold M. The *projective class* of F is a rational n-tuple lying in the solution space of a finite linear system of normal equations defined in terms of the triangulation of M. We refer to this compact, convex linear cell in $\mathbf{R}^n$ as the *projective solution space*. A *vertex surface* in M is a connected, two-sided, normal surface whose projective class is a vertex in the projective solution space. We show that the finite collection of vertex surfaces carries a significant amount of information about the topology of M and a variety of interesting surfaces can always be found among the vertex surfaces. The construction of the vertex surfaces is routine and the results we obtain lead to decision and decomposition algorithms based on procedures using vertex surfaces. Among these algorithms are improvements of earlier algorithms of Haken [H_1], [H_2], and Jaco and Oertel [JO].

The theory of normal surfaces was developed by Haken in the early 1960's and he used it to solve a number of decision problems. In this theory each normal surface F corresponds to a unique integral n-tuple $\mathscr{N}_F$ which is a solution to a finite linear system of matching equations. The normal equations are obtained from these matching equations by the addition of a normalizing equation. The projective class of F is the unit vector in the direction of $\mathscr{N}_F$. A *fundamental surface* is a normal surface whose coordinate $\mathscr{N}_F$ is not the sum of two integral solutions to the matching equations and every normal surface can be obtained as a finite sum of fundamental surfaces. There are only a finite number of fundamental surfaces and these can be found algorithmically. Haken's algorithms are generally based on constructing the set of all fundamental surfaces and looking for surfaces from among this set which shed information on the question being considered. In our algorithms it is the vertex surfaces that provide a source of readily constructed surfaces of significance that can be used to carry out certain decision procedures. While all connected vertex surfaces are either fundamental surfaces or doubles of fundamental surfaces we give examples in §3 that show there are many fundamental surfaces which are not vertex sur-

Received November 13, 1990
1991 Mathematics Subject Classification. Primary 57 M99;
Secondary 57 M10.