ON THE BERGMAN INVARIANT AND CURVATURES OF
THE BERGMAN METRIC

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Introduction

Let $\Omega$ be a (bounded) domain in $\mathbb{C}^n$ and $A^2(\Omega)$ the Bergman space of holomorphic functions in $L^2(\Omega)$. Let $\{\varphi_n\}$ be an orthonormal basis for $A^2(\Omega)$ with respect to the $L^2$-inner product. The Bergman kernel $K_\Omega$ associated to $\Omega$ is given by

$$K_\Omega(z) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(z)}, \quad \forall z \in \Omega.$$  

This kernel, which is independent of the choice of orthonormal basis (see [KR2]), gives rise to an invariant metric, the Bergman metric, on $\Omega$ as follows:

$$B_\Omega(z, u) = \sum_{\alpha, \beta=1}^{n} g_{\alpha\beta}(z)u_\alpha\overline{u_\beta},$$

where $g_{\alpha\beta}(z) = \frac{\partial^2 \log K_\Omega(z)}{\partial z_\alpha \partial \overline{z}_\beta}$. Then the Bergman canonical invariant is defined by

$$J_\Omega(z) \equiv \frac{\det G_\Omega(z)}{K_\Omega(z)}$$

where $G_\Omega(z) = (g_{\alpha\beta}(z))$. Associated with the volume form of the Kähler metric $B_\Omega$ is the Ricci curvature (tensor) $R_\Omega = \sum R_{\alpha\beta}dz_\alpha d\overline{z}_\beta$ which is given by

$$R_{\alpha\beta} = -\frac{\partial^2 \log \det G_\Omega}{\partial z_\alpha \partial \overline{z}_\beta}.$$  

It is easy to see that both $J_\Omega$ and $R_\Omega$ are invariant under biholomorphic mappings. There is an intrinsic relation between these two invariants; details are provided in Proposition 2.1.

The invariant $J_\Omega$ was first introduced by Bergman [BER] to define a general Kähler metric. It is also very useful in deriving various distortion theorems ([BER], [STA]) and in characterizing representative domains [SPR]. The interest in the invariant $J_\Omega$