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ON AN INTEGRAL OPERATOR AND ITS SPECTRUM

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1. Introduction

The action of the differential operator d/dx on the ultraspherical polynomials (spherical harmonics) $C_n^\nu(x)$ is given by

$$\frac{d}{dx} C_n^\nu(x) = 2\nu C_{n-1}^{\nu+1}(x). \tag{1.1}$$

This was used in [6] to provide a right inverse to d/dx. In this note we study the corresponding question for the Pollaczek polynomials $\{P_n^\nu(x; a, b)\}$ [3]. Recall [3] that the Pollaczek polynomials have the generating function

$$\sum_{n=0}^{\infty} P_n^\nu(x; a, b) t^n = (1 - te^{i\theta})^{-\nu+ih(x)}(1 - te^{-i\theta})^{-\nu-ih(x)}, \tag{1.2}$$

with

$$h(x) := \frac{ax + b}{\sqrt{1 - x^2}}, \qquad x = \cos\theta. \tag{1.3}$$

The branch of the square root is the branch that makes $\sqrt{x^2 - 1} \approx x$ as $x \to \infty$. Here

$$e^{i\theta} = x + \sqrt{x^2 - 1}. \tag{1.4}$$

The orthogonality relation of the Pollaczek polynomials is

$$\int_{-1}^{1} P_m^\nu(x; a, b) P_n^\nu(x; a, b) \rho(x; \nu)\, dx = \frac{2\pi \Gamma(n + 2\nu)\delta_{m,n}}{2^{2\nu}(n + a + \nu) n!}, \tag{1.5}$$

and the weight function $\rho(x; \nu)$ is

$$\rho(x; \nu) = (1 - x^2)^{\nu - 1/2} e^{(2\theta - \pi)h(x)} \Gamma(\nu + ih(x))\Gamma(\nu - ih(x)). \tag{1.6}$$

The parameters a, b, ν are assumed to satisfy

$$a > |b| \quad \text{and} \quad \nu > 0. \tag{1.7}$$

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