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# HANKEL OPERATORS ON COMPLEX ELLIPSOIDS

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## 1. Introduction

For $(b_k)$ in $\ell^2 = \ell^2(\mathbb{C})$, the Hankel matrix $H = (h_{k,l})$ is the infinite matrix of which $k, l$ entry is $b_{k+l}$ which may be seen as an operator on $\ell^2$. As it is well known [21], such an operator can be realized as an operator on $H^2(D)$ where $D$ is the unit disc of $\mathbb{C}$: $H^2(D)$ identifies with $\ell^2$ if $(b_k) \in \ell^2$ is identified with $\sum_k b_k z^k$. So, let $b(z) = \sum_k b_k z^k$. Given $f$ in $H^2(D)$, the Hankel operator $h$ is defined by

$$hf = S(b\overline{f}), \tag{1.1}$$

where $S$ is the Szegö projection. Since the family $(z^k)$ is an orthonormal basis of $H^2(D)$, the matrix $H$ and the operator $h$ (see [28]) satisfy

$$(h(z^k)/z^l) = \frac{1}{2i\pi}\int_T b(z)\overline{z}^{k+l}\frac{dz}{z} = b_{k+l} = h_{k,l}.$$

Hankel operators have been studied by many authors. They showed how the properties of the operator or its matrix depend on the symbol $b$. In 1957, Z. Nehari [19] showed that $h$ is bounded if and only if $b$ belongs to $BMO$ and, in 1958, P. Hartman [11] proved that $h$ is a compact operator if and only if $b$ belongs to $VMO$. In 1979, V. V. Peller [20] proved that $h$ is of the Schatten class $\mathcal{S}_p$, $1 \leq p < +\infty$ if and only if $b$ is in the Besov space $B_p^{p,1/p}(D)$. An independent proof was given in 1980 by R. Coifman and R. Rochberg [5] for $p = 1$ and R. Rochberg extended it for $p \geq 1$ [22]. We follow their method.

Let $n \geq 2$ and let $\rho_k = \rho_{k_1,k_2,\dots,k_n}$ be a sequence of positive real numbers. For $b_k$ in the weighted space $\ell^2(\mathbb{C}^n, (\rho_k))$, the generalized Hankel matrix $H = (h_{k,l})$, $(k,l) = ((k_1,\dots,k_n),(l_1,\dots,l_n))$, is the matrix with entries

$$h_{k,l} = b_{k+l}\,\rho_{k+l}.$$

Let $\rho_k = \rho_{k_1,k_2,\dots,k_n} = 1$. We denote by $P^n$ the polydisc in $\mathbb{C}^n$ and by $\partial P^n$ its boundary. The family $e_k(z) = z_1^{k_1}\cdots z_n^{k_n}$ is an orthonornal family of $H^2(P^n)$. Let $b(z) = \sum_k b_k e_k(z)$. The function $b$ is in the Hardy space $H^2(P^n)$ and, again, we can define the Hankel operator $h$ on $H^2(P^n)$ by the relation (1.1). Then we have

$$(h(e_k)/e_l) = \frac{1}{2\pi^n}\int_{\partial P^n} b(\zeta)e_k(\zeta)\overline{e}_l(\zeta)\,d\zeta_1\cdots d\zeta_n = b_{k+l},$$

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