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ON THE $L^{N/2}$-NORM OF SCALAR CURVATURE

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1. Introduction

Let M be a compact n-manifold without boundary. For a Riemannian metric g on M, the curvature tensor, Ricci curvature tensor and scalar curvature of g are denoted by $R(g)$, $\mathrm{Ric}(g)$ and $S(g)$, respectively. A natural and interesting problem in Riemannian geometry is the relations between the topology of the manifold M and curvatures of g. Often the topology of M would impose certain restrictions on the behavior of curvatures of the metric g. The Gauss-Bonnet theorem provides a beautiful relation in this direction. As complexity of the Gauss-Bonnet integrand increases with dimension, it would be desirable to obtain simpler but not "sharp" relations. Indeed, there have been many interests on $L^{\frac{n}{2}}$-curvature pinching and bounds on topological quantities by integral norms of curvatures. In this article, we study some questions on obtaining lower bounds on $L^{\frac{n}{2}}$-norms of the Ricci curvature and scalar curvature. There are some rather general and well-known problems: given a compact n-manifold M, for a sufficiently large class of Riemannian metrics g on M, are there positive lower bounds on

(1) $\mathrm{Vol}\,(M, g)$, provided $K_g \ge -1$ or $\mathrm{Ric}(g)_{ij} \ge -(n-1)g_{ij}$ or $S(g) \ge -n(n-1)$, where K_g is the sectional curvature of (M, g),
(2) $\int_M |S(g)|^{\frac{n}{2}}\, dv_g$ or
(3) $\int_M |\mathrm{Ric}(g)|^{\frac{n}{2}}\, dv_g$?

We note that (2) and (3) are both scale invariant, while a lower bound on curvature is required in (1) so that $\mathrm{Vol}\,(M, g)$ will not go to zero by scaling. As a flat torus would not have positive lower bounds on (1), (2) and (3), some restrictions are needed on the manifold M. Some suggestions are:

(a) M admits a locally symmetric metric of strictly negative sectional curvature;
(b) M admits an Einstein metric of negative sectional curvature;
(c) or simply M admits a metric of negative sectional curvature.

Recently, Besson, Courtois and Gallot [5], [6] have demonstrated that if (M, h) is a compact hyperbolic n-manifold ($n \ge 3$), then for any Riemannian metric g on M with $\mathrm{Ric}(g) \ge -(n-1)g$, one has $\mathrm{Vol}(M, g) \ge \mathrm{Vol}\,(M, h)$ and equality holds if

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