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DEFORMATION CLASSES OF GRADED MODULES AND MAXIMAL BETTI NUMBERS

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1. Introduction

In this paper, I determine the deformation classes of finitely generated graded modules over a polynomial ring $S = k[x_1, \ldots, x_n]$, where k is an infinite field. Theorem 34 states that each deformation class is the set of modules with a given Hilbert function. Furthermore, I show in Theorem 31 that among all quotient modules with a fixed Hilbert function of a given finitely generated graded free module F, the quotient by the lexicographic submodule has the largest graded Betti numbers.

The deformation classes of subschemes of projective space were determined by Hartshorne in his thesis [Ha]. He proved that the Hilbert scheme, $\mathbf{Hilb}^{p(z)}(\mathbb{P}^{n-1})$, is linearly connected. That is, any two subschemes of $\mathbb{P}^{n-1}$ may be deformed to one another if and only if they have the same Hilbert polynomial; if they do, then the deformation may be realized as a sequence of deformations, each defined over $\mathbb{A}^1$. (All deformations in this paper are defined over $\mathbb{A}^1$.) Hartshorne's technique was to construct a deformation from $\mathcal{O}_V = \mathcal{O}_{\mathbb{P}}/\mathcal{I}_V$, the structure sheaf of a subscheme $V \subseteq \mathbb{P}^{n-1}$ with Hilbert polynomial $p(z)$, to $\mathcal{O}_{\mathbb{P}}/\mathcal{J}$, where $\mathcal{J}$ is the sheafification of a "Borel-fixed" ideal. Then, he constructed special families called "fans" which give a sequence of deformations between any two such $\mathcal{O}_{\mathbb{P}}/\mathcal{J}$ with Hilbert polynomial $p(z)$.

Reeves, in her thesis [Re1,2], refined Hartshorne's techniques in characteristic zero and showed that if d is the degree of $p(z)$, then there is a sequence of no more than $d+2$ deformations defined over $\mathbb{A}^1$ taking $\mathcal{O}_{\mathbb{P}}/\mathcal{I}_V$ to $\mathcal{O}_{\mathbb{P}}/\mathcal{L}$, where $\mathcal{L}$ is the sheafification of the unique "lexicographic ideal" L such that S/L has Hilbert polynomial $p(z)$, and has no submodule of finite length. This is the essential point in her theorem on the radius of the Hilbert scheme.

The main technique in this paper is a refinement of the technique that Reeves used in her thesis. Indeed, the operation that I call Φ in this paper is the essential operation in her argument. On the way to proving the two main theorems of this paper, I will show that Reeves' bound of $d + 2$ holds in positive characteristic, and also for deformations of quotient sheaves of a sum of line bundles $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-d_i)$. In particular, the quot scheme $\mathbf{Quot}^{p(z)}(\mathcal{E})$ is linearly connected for such an $\mathcal{E}$.

Lexicographic submodules of a free module F play a central role in this paper, and I will now describe them. Let $S = k[x_1, \ldots, x_n]$ where k is a field and let F be a

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