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NORM INEQUALITIES IN THE CORACH-PORTA-RECHT THEORY AND OPERATOR MEANS

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1. Introduction

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space. In particular, an operator A on H is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$.

In [2], Corach-Porta-Recht gave a norm inequality as a key of their theory on differential geometry. Afterwards, we pointed out that it is equivalent to the Heinz inequality [6]. On the other hand, Furuta [10] showed that the Cordes inequality

$$\|A^t B^t\| \leq \|AB\|^t \quad \text{for } A, B \geq 0 \text{ and } 0 \leq t \leq 1 \tag{1}$$

is equivalent to the Löwner-Heinz inequality (cf. [16])

$$A \geq B \geq 0 \quad \text{implies} \quad A^t \geq B^t \quad \text{for } 0 \leq t \leq 1. \tag{2}$$

Under such situation, we developed Furuta's argument on the equivalence of (1) and (2) in [8]. However the Jensen inequality [12]

$$(X^* A X)^t \geq X^* A^t X \quad \text{for } A \geq 0 \text{ and contractions } X \tag{3}$$

is not discussed there.

Very recently, Corach-Porta-Recht [3] proposed the norm inequality, denoted the CPR inequality,

$$\|(A \sharp_t B)^{1/2} (C \sharp_t D)^{1/2}\| \leq \|A^{1/2} C^{1/2}\|^{1-t} \|B^{1/2} D^{1/2}\|^t \tag{4}$$

for positive operators A, B, C and D, where $\sharp_t$ is the t-power mean defined by

$$A \sharp_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2} \tag{5}$$

for invertible $A, B \geq 0$ and $t \in [0, 1]$; see [15]. As stated in [3], (1) is the special case of (4), i.e., take $A = C = 1$ in (4). For the sake of convenience, the t-power mean defined by (5) is extended as in [11]: For $t \in \mathbb{R}$,

$$A \natural_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2} \tag{5'}$$

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