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UMBILIC FOLIATIONS AND CURVATURE

GERARD WALSCHAP

Recall that a submanifold L of a Riemannian manifold is totally umbilic if it curves equally in all directions, i.e., if there is some vector $N \perp L$ such that the second fundamental tensor S_X of L in any direction $X \perp L$ is given by $S_X = \langle X, N \rangle \operatorname{Id}$. In this note, we investigate some properties of k-dimensional Riemannian foliations $\mathcal{F}^k$ with totally umbilic leaves, which we call umbilic foliations for short. Notice that a Riemannian flow ($k = 1$) is always umbilic. It is to be expected that restrictions increase with k. In fact, we show that on manifolds M^n of positive sectional curvature, there are no umbilic foliations if $k > (n-1)/2$. This is a best possible estimate, since for example a Euclidean 3-sphere admits an abundance of Riemannian flows. Similarly, it turns out that on spaces of nonpositive curvature, an umbilic foliation of dimension $n-1$ is, when lifted to the universal cover, 'almost always' a foliation by horospheres or by hypersurfaces equidistant from a totally geodesic one.

Although Riemannian foliations $\mathcal{F}^k$ seem to arise naturally in geometry (see for instance [12] for a recent spectacular example), our understanding of them is still quite limited, even in the simplest case, that of constant curvature: Gromoll and Grove have shown that on Euclidean spheres they are homogeneous—i.e., orbit foliations of groups of isometries—if $k \leq 3$ [8], and that Riemannian flows are always flat (the orthogonal complement is an integrable totally geodesic distribution) or homogeneous in any space of constant curvature [7]. We show that the latter result extends to all umbilic foliations on space forms. The starting point is the observation that on manifolds with curvature bounded from below, the mean curvature form of an umbilic foliation is closed as soon as it is basic (this also generalizes some results in [5], [6], [7]). As a consequence, on such manifolds, Riemannian flows with basic mean curvature are necessarily homogeneous; i.e., they are locally spanned by a Killing field.

1. The local geometry

For notational purposes, we begin by recalling some elementary facts about Riemannian foliations. The reader is referred to [2], [8], [11] for further details and other

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