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THE CENTRALIZER OF INVARIANT FUNCTIONS AND DIVISION PROPERTIES OF THE MOMENT MAP

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1. Introduction

Let $\Phi\colon M \longrightarrow \mathfrak{g}^*$ be a moment map associated to a Hamiltonian action of a compact connected Lie group G on a compact [1] connected symplectic manifold (M, ω). Pullbacks by Φ of smooth functions on $\mathfrak{g}^*$ are called *collective functions*. They form a Poisson subalgebra of the algebra of smooth functions on M. Its centralizer is the algebra of invariant functions; i.e., a smooth function f on M is invariant if and only if $\{f, h\} = 0$ for every collective function h, where $\{\ ,\ \}$ denotes the Poisson bracket corresponding to the symplectic form ω.

Motivated by a study of completely integrable systems in [GS1], Guillemin and Sternberg conjectured in [GS3] that the centralizer of the algebra of invariant functions is the algebra of collective functions. They proved this conjecture for neighborhoods of generic points in M.

A collective function is clearly constant on the level sets of the moment map. The converse need not be true. For example, the standard linear action of the group $G = SU(2)$ on $\mathbb{C}^2$ has a moment map $\Phi(u, v) = (\overline{u}v, \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2)$ when we identify the vector space $\mathfrak{g}^*$ with $\mathbb{R} \times \mathbb{C}$. The function $f(u, v) = |u|^2 + |v|^2$ is constant on the level sets of Φ because it is equal to $(|\overline{u}v|^2 + (\frac{1}{2}|u|^2 - \frac{1}{2}|v|^2)^2)^{\frac{1}{2}} = 2||\Phi||$. It is not collective because the function $||x||$ is not smooth on $\mathbb{R} \times \mathbb{C}$.

In Section 2 of this paper we show that the centralizer of the algebra of invariant functions is the algebra of functions that are constant on the level sets of the moment map. In fact, these two algebras are mutual centralizers in the Poisson algebra $C^\infty(M)$. See Theorem 1 and Corollary 2.12. This was already shown in the thesis of the first author [K], but our current proof is shorter.

This result raises the following question: what is the obstruction for a function that is constant on the level sets of the moment map to be collective? In Section 3, Theorem 2, we express this obstruction as a condition on the Taylor series of the function. The proof uses theorems of Bierstone and Milman and of Marle, Guillemin, and Sternberg. Theorem 2 essentially reduces the identification of the centralizer of the invariant functions to an algebraic question. Based on this, F. Knop recently

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[1] Throughout this introduction we assume that the manifold M is compact and the group G is connected. In the rest of the paper our assumptions are often more general.

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