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VARIATION IN PROBABILITY, ERGODIC THEORY AND ANALYSIS

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1. Introduction

In many areas of analysis, ergodic theory and probability, square functions have proven to be one of the most useful tools to study convergence properties. (See the paper by Stein [17] for a very informative historical discussion of the importance of various square functions in several areas.) For example, the martingale square function was used by Burkholder, Gundy and Silverstein [7] to give the first real variable characterization of H_p. An ergodic square function was used by Bourgain [3] in his proof that the ergodic averages along the sequence of squares converge a.e. In this paper we consider operators that are closely related to the square functions, but have very different properties.

Let $(\mathcal{F}_k)$ denote an increasing sequence of σ-fields. Then the martingale square function is defined by

$$Sf(x) = \left(\sum_{k=1}^{\infty} |E_k f(x) - E_{k-1} f(x)|^2\right)^{\frac{1}{2}},$$

where E_k denotes the conditional expectation operator with respect to the σ-field $\mathcal{F}_k$. This operator, which maps L^p to L^p for each p, $1 < p < \infty$, gives a measure of the square variation of the martingale sequence $(E_k f)$. It is natural to ask about the L^p boundedness properties of the q-variation operator

$$V_q f(x) = \left(\sum_{k=1}^{\infty} |E_k f(x) - E_{k-1} f(x)|^q\right)^{\frac{1}{q}},$$

for $1 \leq q < 2$. In Section 2 we show that if $q < 2$ then the operator V_q is very badly behaved. In particular, we show that it is possible to have $V_q f(x) = \infty$ a.e. even for bounded functions, f. The arguments provide a revealing contrast to the Hilbert space techniques that come into play when $q \geq 2$. The martingale result is

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