ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 1, Spring 1998

NEST ALGEBRAS ARE HYPERFINITE

KENNETH R. DAVIDSON

In [12], Paulsen, Power and Ward show that nest algebras are semidiscrete. This is an important tool in developing a good dilation theory for representations of nest algebras (see also [13]). These ideas have been extended to establish semidiscreteness and dilation theorems for larger classes of nonself-adjoint operator algebras [6], [4]. Paulsen and Power have asked whether nest algebras actually have the stronger property of hyperfiniteness. In this paper, we establish this via a refinement of the techniques used in [12] and [5].

A weakly closed operator algebra in a category $\mathcal{C}$ is *hyperfinite* if it is the increasing union of finite dimensional subalgebras which are completely isometrically isomorphic to (finite dimensional) members of $\mathcal{C}$. For von Neumann algebras, deep results of Connes, Haagerup, Choi, Effros and others have shown that hyperfiniteness is equivalent to various other properties including semidiscreteness and amenability. Moreover hyperfiniteness is a stronger condition in the sense that it readily implies the others for elementary reasons.

Paulsen, Power and Ward show that for any nest algebra $\mathcal{T}(\mathcal{N})$ on a separable Hilbert space, there is a sequence $\mathcal{A}_n$ of finite dimensional nest algebras together with completely isometric homomorphisms Φ_n of $\mathcal{A}_n$ into $\mathcal{T}(\mathcal{N})$ and completely contractive weak-$*$ continuous maps E_n of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{A}_n$ such that $\Psi_n = \Phi_n E_n$ are idempotent maps converging point–weak-$*$ to the identity on $\mathcal{T}(\mathcal{N})$ and converging in norm on $\mathcal{T}(\mathcal{N}) \cap \mathcal{K}$, where $\mathcal{K}$ is the ideal of compact operators. In our argument, we achieve this but in addition arrange that the algebras $\mathcal{B}_n = \Phi_n(\mathcal{A}_n)$ are nested unital algebras.

An even stronger form of hyperfiniteness would require the imbeddings α_n of $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ induced by the containment of $\mathcal{B}_n$ in $\mathcal{B}_{n+1}$ to be nice maps. Recent interest has been focussed on imbeddings which extend to $*$-endomorphisms of the enveloping matrix algebras $\mathfrak{A}_n$ (isomorphic to the $k \times k$ matrices $\mathfrak{M}_k$ for some k) which are regular in the following sense. The algebras $\mathcal{A}_n$ each contain a masa $\mathcal{D}_n$ of $\mathfrak{A}_n$ which form an increasing sequence. They determine a set of matrix units for each matrix algebra $\mathfrak{A}_n$; and $\mathcal{A}_n$ are block upper triangular with respect to this basis. The imbedding is *regular* if each matrix unit of $\mathfrak{A}_n$ is sent to a sum of matrix units in $\mathfrak{A}_{n+1}$. The direct limit of the sequence $(\mathcal{A}_n, \alpha_n)$ is a subalgebra $\mathcal{A}$ of the AF C*-algebra

Received February 12, 1997.
1991 Mathematics Subject Classification 47D25.
The author was partially supported by an NSERC grant and a Killam Research Fellowship.