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# CAUCHY TRANSFORMS AND COMPOSITION OPERATORS

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**1.** A holomorphic function $f$ on the unit disk $\mathbb{D}$ is a Cauchy transform if it admits a representation

$$f(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \bar{\zeta} z}, \tag{1.1}$$

where $\mu$ is a finite Borel measure on the unit circle $\mathbb{T}$. The space $K$ of all Cauchy transforms becomes a Banach space under the norm $\|f\|_K = \inf \|\mu\|_M$, where the infimum is taken over all Borel measures $\mu$ satisfying (1.1). The Banach space $K$ is clearly the quotient of the Banach space $M$ of Borel measures by the subspace of measures with vanishing Cauchy transforms. It is an immediate consequence of the F. and M. Riesz theorem that a Borel measure $\mu$ has a vanishing Cauchy transform if and only if $\mu$ has the form $d\mu = f\,dm$, where $f \in \bar{H}_0^1$ and $m$ is normalized Lebesgue measure on $\mathbb{T}$. Here $\bar{H}_0^1$ is the subspace of $L^1$ consisting of functions with mean value 0 whose conjugates belong to the Hardy space $H^1$. Hence $K$ is isometrically isomorphic to $M/\bar{H}_0^1$. On the other hand, $M$ admits a decomposition $M = L^1 \oplus M_s$, where $M_s$ is the space of Borel measures which are singular with respect to Lebesgue measure, and $\bar{H}_0^1 \subset L^1$. Consequently $K$ is isometrically isomorphic to $L^1/\bar{H}_0^1 \oplus M_s$. In particular $K$ admits an analogous decomposition $K = K_a \oplus K_s$, where $K_a$ is isometrically isomorphic to $L^1/\bar{H}_0^1$ and $K_s$ to $M_s$.

Now let $\phi$ be a holomorphic map of the unit disk $\mathbb{D}$ into itself. The composition operator $C_\phi f = f \circ \phi$ acts on a variety of spaces of holomorphic functions, most notably the Hardy spaces $H^p$. It was established by Bourdon and Cima [2] that $C_\phi$ also acts on $K$, that is, $f \circ \phi \in K$ for all $f \in K$. It is an immediate consequence of the closed graph theorem that $C_\phi$ is a bounded operator on $K$. In fact Bourdon and Cima provide the estimate

$$\|C_\phi\|_K \leq \frac{2 + 2\sqrt{2}}{1 - |\phi(0)|} \tag{1.2}$$

for the norm of $C_\phi$ on $K$. A new proof of the boundedness of $C_\phi$ on $K$ will follow

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