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EIGENVALUES OF LAPLACIANS WITH MIXED BOUNDARY CONDITIONS, UNDER CONFORMAL MAPPING

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1. Introduction

The prototypical result of this paper says, roughly, that if $f(z) = \sum_{j\in\mathbf{Z}} a_j z^j$ is a conformal map of an annulus A onto a doubly connected plane domain Ω with $|a_1| = 1$, then

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j(\Omega)^s} \geq \sum_{j=1}^{\infty} \frac{1}{\lambda_j(A)^s} \qquad \text{for all } s > 1,$$

where $\lambda_j(\Omega)$ is the j-th eigenvalue of the Laplacian on Ω under Dirichlet boundary conditions on the outer boundary of Ω and Neumann conditions on the inner boundary, and similarly for $\lambda_j(A)$. That is, the zeta function of the Laplacian is at least as big for Ω as it is for the annulus A.

This introduction provides some historical context; then in Section 2 the results are all stated precisely. For similar results but under purely Dirichlet boundary conditions, see the earlier paper [13], written with C. Morpurgo. This present work draws heavily on the arguments and intuition in [13], and is best read in conjunction with that paper.

The eigenvalues of the Laplacian have many physical interpretations, for example as the frequencies of vibration of a membrane, as rates of decay for the heat (or mass diffusion) equation, and as cut-off frequencies for waveguides. However, the eigenvalues of doubly connected regions can be calculated exactly only for a few special regions, most notably for annuli, and while numerical methods are sophisticated and successful [11], they can only estimate finitely many of the eigenvalues. This paper will give sharp estimates involving *all* the eigenvalues. Incidentally, the mixed boundary conditions employed in this paper have drawn increasing attention in recent years (see for example [4], [17] and the references therein).

G. Pólya and G. Szegő [16] proved by conformal transplantation an *upper* bound on the first eigenvalue of a simply connected plane domain under Dirichlet boundary conditions: if $f(z)$ is a conformal map of the open unit disk D onto a bounded, simply connected plane domain Ω and if $|f'(0)| = 1$, then $\lambda_1(\Omega) \leq \lambda_1(D)$. In [13, Cor. 3], the author and C. Morpurgo proved a direct analogue of this for doubly connected

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