UNIVERSAL LOCALLY CONNECTED REFINEMENTS

BY

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We shall prove the existence of a locally connected space associated with an arbitrary topological space which has a universal property with respect to maps of locally connected spaces. We shall also obtain a similar result for uniform spaces.

The proofs of these results are much easier when the notation emphasizes that a topological space is a set together with a definite family of subsets. However, for applications it is convenient to have the theorems in the usual language; hence we shall now state our results in such terms.

THEOREM A. *Let* $\mathbf{S}$ *be a topological space. There exist a locally connected topological space* $\mathbf{S}^*$ *and a continuous one-to-one mapping* φ *of* $\mathbf{S}^*$ *onto* $\mathbf{S}$ *such that*

> *If* $\mathbf{f}$ *is any continuous mapping of a locally connected space* $\mathbf{A}$ *into* $\mathbf{S}$, *then* $\mathbf{f}$ *can be factored in the form* $\mathbf{f} = \varphi \circ \mathbf{f}^*$ *where* $\mathbf{f}^*$ *is a continuous mapping of* $\mathbf{A}$ *into* $\mathbf{S}^*$.

The pair $\langle \mathbf{S}^*, \varphi \rangle$ *is unique within isomorphism.*

A special case of this theorem, used in [1, pp. 54–55], provided the motivation for this work. A similar theorem for locally arcwise connected spaces is proved in [2].

The author is indebted to the referee for calling his attention to a paper of G. S. Young [5] in which many similar theorems are established. In particular the *c*-topology of Young is our $\mathfrak{T}'$, the *a*-topology is the topology of [2], and the *lc*-topology appears to be in general intermediate between the $\mathfrak{T}^*$-topology and the *a*-topology. The two latter coincide for complete metric spaces.

We shall refer to the pair $\langle \mathbf{S}^*, \varphi \rangle$ of Theorem A as a universal locally connected refinement of $\mathbf{S}$.

THEOREM B. *Let* $\mathbf{S}$ *be a topological space, and let* $\langle \mathbf{S}^*, \varphi \rangle$ *be a universal locally connected refinement of* $\mathbf{S}$. *If* $\mathbf{S}$ *satisfies the first axiom of countability or any of the separation axioms* T_0, T_1, T_2, *or* T_3, *then so does* $\mathbf{S}^*$. *If the topology of* $\mathbf{S}$ *can be derived from a uniform structure or a metric, then the same is true of* $\mathbf{S}^*$; *hence* $\mathbf{S}^*$ *will be completely regular whenever* $\mathbf{S}$ *is.*

On the other hand, if $\mathbf{S}$ is separable, compact, locally compact, paracompact, or connected, then $\mathbf{S}^*$ need not have these properties. Similarly the separation axiom T_4 need not transfer from $\mathbf{S}$ to $\mathbf{S}^*$, but we do not answer the question concerning T_5.

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