A CONVERSE TO THE DOMINATED CONVERGENCE THEOREM

BY

David Blackwell[1] and Lester E. Dubins[2]

1. Introduction and summary

On a probability space $(\Omega, \mathcal{B}, P)$, let $\{f_n, n = 1, 2, \cdots\}$ be a sequence of nonnegative random variables in L_1 such that $f_n \to f \in L_1$ with probability 1, and define $g = \sup_n f_n$. If $g \in L_1$, the Lebesgue dominated convergence theorem asserts that $E(f_n) \to E(f)$. More generally, as noted by Doob [1, p. 23], if $g \in L_1$, then for any Borel field $\mathcal{B}_0$ contained in $\mathcal{B}$,

$$E(f_n \mid \mathcal{B}_0) \to E(f \mid \mathcal{B}_0) \quad \text{a.e.} \tag{1}$$

If one extends this result in a minor manner, Lebesgue's condition $g \in L_1$ is not only sufficient but necessary, as the following converse to the dominated convergence theorem asserts.

Theorem 1. *If $f_n \geqq 0$, $f_n \to f$ a.e., $f_n \in L_1$, $f \in L_1$, and $g = \sup_n f_n \notin L_1$, there are, on a suitable probability space, random variables $\{f_n^*, n = 1, 2, \cdots\}, f^*$, and a Borel field $\mathcal{C}$ such that $f^*, f_1^*, f_2^*, \cdots$ have the same joint distribution as $f, f_1, f_2, \cdots$, and*

$$P\{E(f_n^* \mid \mathcal{C}) \to E(f^* \mid \mathcal{C})\} = 0. \tag{2}$$

In view of this result, it is of interest to find conditions which will ensure that $g \in L_1$. As a special case of interest, let h be a nonnegative random variable in L_1, let $\mathcal{B}_n$ be a monotone sequence of Borel fields contained in $\mathcal{B}$, and let $f_n = E(h \mid \mathcal{B}_n)$. Doob [1, p. 317] has shown that if $h \log h \in L_1$, then also $g = \sup_n f_n \in L_1$. It turns out that the condition $h \log h \in L_1$ is necessary, as well as sufficient, in the following sense:

Theorem 2. *If $h \geqq 0$, $h \in L_1$, $h \log h \notin L_1$, there are, on a suitable probability space, a random variable h^* with the same distribution as h and a monotone sequence $\mathcal{B}_n^*$ of Borel fields, which can be chosen either increasing or decreasing, for which*

$$g^* = \sup_n E(h^* \mid \mathcal{B}_n^*) \notin L_1. \tag{3}$$

Theorem 2 will be an immediate consequence of the following result, which gives sharp upper bounds on the distribution of g^*, rather than only information about the expectation of g^* as in Theorem 2.

Theorem 3. *Let h^* be any nonnegative random variable in L_1, and let h be the (essentially unique) nonincreasing function on the unit interval $(0, 1]$ whose*

Received March 19, 1962.

[1] This paper was prepared with the partial support of the Office of Naval Research.

[2] Prepared with the partial support of the National Science Foundation.