GEODESIC SPHERES IN GRASSMANN MANIFOLDS

BY

JOSEPH A. WOLF[1]

1. Introduction

Let $\mathbf{G}_{n,k}(\mathbf{F})$ denote the Grassmann manifold consisting of all n-dimensional subspaces of a left k-dimensional hermitian vectorspace $\mathbf{F}^k$, where $\mathbf{F}$ is the real number field, the complex number field, or the algebra of real quaternions. We view $\mathbf{G}_{n,k}(\mathbf{F})$ as a Riemannian symmetric space in the usual way, and study the connected totally geodesic submanifolds $\mathbf{B}$ in which any two distinct elements have zero intersection as subspaces of $\mathbf{F}^k$. Our main result (Theorem 4 in §8) states that the submanifold $\mathbf{B}$ is a compact Riemannian symmetric space of rank one, and gives the conditions under which it is a sphere. The rest of the paper is devoted to the classification (up to a global isometry of $\mathbf{G}_{n,k}(\mathbf{F})$) of those submanifolds $\mathbf{B}$ which are isometric to spheres (Theorem 8 in §13). If $\mathbf{B}$ is not a sphere, then it is a real, complex, or quaternionic projective space, or the Cayley projective plane; these submanifolds will be studied in a later paper [11].

The key to this study is the observation that any two elements of $\mathbf{B}$, viewed as subspaces of $\mathbf{F}^k$, are at a constant angle (*isoclinic* in the sense of Y.-C. Wong [12]). Chapter I is concerned with sets of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^{2n}$, and we are able to extend Wong's structure theorem for such sets [12, Theorem 3.2, p. 25] to the complex numbers and the quaternions, giving a unified and basis-free treatment (Theorem 1 in §4). Essentially, we introduce a "closure" operation on the collection of all such sets, and characterize the "closed" sets by means of linear transformations which satisfy some equations studied by A. Hurwitz [6] in connection with quadratic forms permitting composition. We give the closed sets the name *isoclinic sphere*; the first result of Chapter II is that an isoclinic sphere on $\mathbf{F}^{2n}$ is a totally geodesic submanifold of $\mathbf{G}_{n,2n}(\mathbf{F})$ which is isometric to a sphere (Theorem 2 in §6). We then prove a strong converse (Theorem 3 in §7) which allows us to prove our main result (Theorem 4 in §8) by reducing it to the case where $\mathbf{B}$ is an isoclinic sphere on a $2n$-dimensional subspace of $\mathbf{F}^k$.

Chapter III is devoted to the classification of isoclinic spheres on $2n$-dimensional subspaces of $\mathbf{F}^k$, up to equivalence under the full group of isometries of $\mathbf{G}_{n,k}(\mathbf{F})$. We first consider the case $k = 2n$. Our structure theorem for isoclinic spheres (Theorem 1) shows that isoclinic spheres can all be obtained from certain representations of Clifford algebras. Sections 10 and 11 are devoted to the study of these representations, and yield (Theorem 6 in §11)

Received January 29, 1962.

[1] The author wishes to thank the National Science Foundation for fellowship support during the period of preparation of this paper.