PROBABILITY DISTRIBUTIONS ON LOCALLY COMPACT ABELIAN GROUPS

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1. Introduction

For probability distributions on the real line there are three main theorems on which the entire study of limit theorems for sums of independent random variables is based. These are (1) the Lévy-Khinchin representation of an infinitely divisible distribution, (2) the criteria for weak convergence of such distributions, and (3) Khinchin's theorem on sums of infinitesimal summands stating that these converge weakly if and only if certain associated infinitely divisible laws converge. For a precise statement of these results we refer to Kolmogorov and Gnedenko [3].

During the last two decades or so these results have been extended by many authors to varying degrees of generality. We mention in particular the works of Lévy [12], Kawada and Itô [17], Takano [9], Bochner [1], [2], Hunt [4], Urbanik [13], [14], Kloss [16]. In this paper we study probability distributions on a locally compact abelian (separable) group and obtain definitive extensions of all the three main results mentioned above.

The preliminaries are developed in Section 2. We mention in particular the concept of shift-compactness introduced therein and the important role that Theorem 2.1 plays in our study. A slightly modified notion of an infinitely divisible law is given in this paper to take into account the fact that the group may not be divisible.

The main results of the paper are the following. Weak limits of sums of uniformly infinitesimal random variables (with values in a group) are infinitely divisible. These limits can be obtained from certain accompanying infinitely divisible distributions if they have no idempotent factors. If μ is any infinitely divisible distribution without an idempotent factor, then its characteristic functional $\hat{\mu}(y)$, defined on the character group, has the form

$$(x_0, y) \exp\left\{\int [(x, y) - 1 - ig(x, y)] \, dF(x) - \phi(y)\right\}$$

where (x, y) is the value of the character y at x, x_0 is a fixed element of the group, $g(x, y)$ is a fixed function independent of μ, F is a σ-finite measure which integrates the function $(x, y) - 1 - ig(x, y)$ and has finite mass outside each neighborhood of the identity, and $\phi(y)$ is a nonnegative continuous function satisfying the equality

$$\phi(y_1 + y_2) + \phi(y_1 - y_2) = 2[\phi(y_1) + \phi(y_2)].$$

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