INTEGRAL EQUATIONS AND SEMIGROUPS[1]

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Consider an $n \times n$ matrix ϕ of absolutely continuous functions on an interval S of real numbers. If each of G and H is also such a matrix, then, as is well known (cf. [2, p. 352] for comments and references), the differential requirement that

$$\left.\begin{aligned} G'(s) - G(s)\phi'(s) &= 0 \\ H'(s) + \phi'(s)H(s) &= 0 \end{aligned}\right\} \text{ almost everywhere on } S,$$

where 0 is the $n \times n$ zero matrix, is equivalent to the (Stieltjes) integral requirement that if c is in S then for all x and y in S

$$G(y) = G(c) + \int_c^y G\cdot d\phi \quad \text{and} \quad H(x) = H(c) + \int_x^c d\phi\cdot H;$$

moreover, there is a fundamental matrix W of continuous functions on $S \times S$ which satisfies—without exception—

$$\text{(i)} \qquad W(x, y) = 1 + \int_x^y W(x, \ )\cdot d\phi = 1 + \int_x^y d\phi\cdot W(\ , y)$$

where 1 is the $n \times n$ unit matrix, and provides G and H in the form

$$G(y) = G(c)W(c, y) \quad \text{and} \quad H(x) = W(x, c)H(c).$$

The relationship (i) has been extended by H. S. Wall [9], [10], with the condition of absolute continuity on ϕ replaced by that of continuity and bounded variation, the intrinsic nature of the *harmonic matrices* W so obtained being determined explicitly; the reciprocal formulas (involving sum- and product-integrals)

$$\text{(ii)} \quad \phi(y) - \phi(x) = \int_x^y W(\ , c)\cdot dW(c, \ ), \qquad W(x, y) = {}_x\prod^y[1 + d\phi]$$

were discovered, respectively, by Wall [10] and this author [4]. The continuity condition on ϕ has been relaxed, in two different directions, by T. H. Hildebrandt [2] and by the present author [5], [6].

This paper is concerned with connections between additive and multiplicative integration processes, where the integration is directed along intervals in some linearly ordered system and the functions involved satisfy various conditions of boundedness, having their values in a normed algebraic ring which is complete as a metric space.

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