DISJOINT PAIRS OF SETS AND INCIDENCE MATRICES

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In a recent paper [1] investigating the repeated appearance of zeros in the powers of a matrix the following purely combinatorial problem arose. Let $x_1, \cdots, x_n$ be n distinct objects, and let $S_1, \cdots, S_n$ be n subsets of these objects satisfying the following two conditions:

(a) For no s, $s = 1, \cdots, n - 1$, does the union of some s of the sets $S_1, \cdots, S_n$ contain s or fewer elements.

(b) No two of the sets S_i intersect in precisely one of the x_j.

Question: What is the maximum possible number $w(n)$ of nonintersecting pairs of sets, S_p and S_q, $1 \leqq p < q \leqq n$?

As usual we reformulate the problem in terms of the incidence matrix of the configuration: let A be an n-square $(0, 1)$-matrix whose (i, j) entry is 1 or 0 according as x_j belongs to S_i or not. The conditions (a) and (b) simply state respectively that A has no $s \times (n - s)$ submatrix of zeros, $s = 1, \cdots, n - 1$, and AA' has no entry equal to 1.

An n-square matrix is said to be *partly decomposable* if it contains an $s \times (n - s)$ zero submatrix for some s. Otherwise it is called *fully indecomposable.* Let $\Omega(n)$ denote the totality of fully indecomposable n-square $(0, 1)$-matrices such that AA' contains no entries equal to 1. Let $z(M)$ denote the number of zeros in a matrix M. Then the number of zeros above the main diagonal in AA', where A is fully indecomposable, is $z(AA')/2$. In our problem we consider the number

$$w(n) = \max_{A \in \Omega(n)} z(AA')/2,$$

i.e., the maximum number of zeros above the main diagonal in AA' as A varies over $\Omega(n)$. Marcus and May obtained in [1] the following results:

$$w(2) = w(3) = 0, \qquad w(4) = 1, \qquad w(5) = 2;$$

$$w(n) < n(n - 3)/2 \qquad \textit{for } n \geqq 4;$$

$$w(n) \geqq n(n - 6)/2 \qquad \textit{if } n \textit{ is even},$$

$$w(n) \geqq (n(n - 6) - 3)/2 \quad \textit{if } n \textit{ is odd}.$$

The main result of the present paper is Theorem 6 which states:

$$w(n) = n(n - 4)/2 \qquad \textit{if } n \textit{ is even and } n \geqq 6,$$

$$w(n) = (n(n - 4) - 3)/2 \quad \textit{if } n \textit{ is odd and } n \geqq 7.$$

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