ON THE HOMOLOGY DECOMPOSITION OF POLYHEDRA[1]

BY

C. R. CURJEL

Introduction. Summary

Let X_1 be a simply connected polyhedron (i.e., a simply connected finite CW-complex). According to Eckmann-Hilton [6]—see also Brown-Copeland [3]—it is homotopy-equivalent to a polyhedron X built up by subcomplexes X_i, $X_2 \subset X_3 \subset \cdots \subset X_N = X$, where X_r is constructed out of X_{r-1} in a very perspicuous way by means of the r^{th} integer homology group of X and an element in a homotopy group of X_{r-1}. Following [6] we call $X = \{X_r\}$ a normal polyhedron, and the collection $\{X_r\}$ of the X_r a homology decomposition of X.

It is the purpose of this note to exemplify our opinion that the concept of the homology decomposition can be used profitably to study homotopy sets $\Pi(X, Y)$ of the maps of a space X into a space Y.

All considerations rely on Proposition 2.2 which describes the circumstances under which a map $f : X \to Y$ of the normal polyhedra $X = \{X_r\}$, $Y = \{Y_r\}$ induces a map $f_r : X_r \to Y_r$ compatible with f. Proposition 2.2 follows from Proposition 2.1, which generalizes the Blakers-Massey theorem on relative homotopy groups [2, p. 198].

Section 3 contains the first example of an application of the homology decomposition. Proposition 3.3 is a powerful lemma of Thom [10, p. 59], for which we give a new proof. The idea of our proof is to climb up a homology decomposition, using at each step known facts about homotopy groups of spheres.

From Section 4 on, we restrict our attention to "selfmaps" $f : X \to X$ of a simply connected polyhedron X. The composition of maps defines in the homotopy set $\Pi(X, X)$ a multiplication turning $\Pi(X, X)$ into a monoid. Denote by $T(X)$ the homotopy set of all selfmaps of X which induce the trivial endomorphism of $\oplus_k H^k(X; H_k(X))$. It is a multiplicatively closed subset of $\Pi(X, X)$. Theorem 4.2 states that $T(X)$ is nilpotent. The order $t(X)$ of nilpotency of $T(X)$ is a homotopy invariant of X which, by appealing to a theorem of Novikov [8], can be shown to assume any given value for an appropriate X (Proposition 4.5).

An endomorphism Φ of $\oplus_k H^k(X; H_k(X))$ induced by a map $f : X \to X$ satisfies necessarily a certain relation, and such a Φ will be called admissible (Definition 4.6, Lemma 4.7). The question for which spaces X every admissible endomorphism of $\oplus_k H^k(X; H_k(X))$ can be realized by a selfmap of X is dealt with in Theorem 4.9: For 2-connected X this is the case if and

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