# SUZUKI 2-GROUPS

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## 1. Introduction

In this paper we shall determine all groups $G$ of order a power of 2 which possess automorphisms $\xi$ that permute their involutions cyclically. The determination is complete, except that we do not exclude the possibility that two or more of the groups that we list may be isomorphic. The investigation is perhaps not without interest simply as an example of the use of linear methods in $p$-group theory; but the main motivation for it is that some result along these lines is needed by Suzuki in his classification [4] of $ZT$-groups. It is a pleasure to acknowledge that this paper is, in a direct way, a fruit of the special year in Group Theory organized by the Department of Mathematics at the University of Chicago.[1]

A 2-group with only one involution, that is, a cyclic or generalised quaternion group obviously has the property under discussion; and an abelian group has it if and only if it is a direct product of cyclic 2-groups all of the same order. It is convenient to exclude these cases from the beginning, and define a *Suzuki 2-group* as a non-abelian 2-group with more than one involution, having a cyclic group of automorphisms which permutes its involutions transitively.

Evidently, the involutions of a Suzuki 2-group $G$ all belong to its center, and so constitute, with the identity, an elementary abelian subgroup $\Omega_1(G)$ of order $q = 2^n$, $n > 1$. We shall show that $\Omega_1(G) = Z(G) = \Phi(G) = G'$, so that $G$ is of exponent 4 and class 2. The automorphism $\xi$ which permutes cyclically the $q - 1$ involutions evidently has order divisible by $q - 1$. We shall show that $\xi$ can be taken to have order precisely $q - 1$, and so to be regular. The order of $G$ is either $q^2$ or $q^3$.

In many ways, it would be more satisfactory to impose on $G$ the simpler, weaker condition that the involutions of $G$ are permuted transitively by the full automorphism group of $G$. Possibly such a relaxation would not bring in any large class of new groups; but the condition seems to be very hard to handle. However, a little of our argument extends to the general case, and this part has been stated for that case.

The methods used are similar to those involving the associated Lie ring (cf. e.g. [2]), but we shall not construct this ring explicitly. The setup, which we shall presuppose, is as follows. If $H$ is a subgroup of the 2-group $G$, and $K$ a normal subgroup of $H$ with elementary abelian factor group $H/K$,

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