BIG PROJECTIVE MODULES ARE FREE[1]

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1. Introduction

Finitely generated projective modules arise significantly in certain geometric and arithmetic questions. We shall show here that nonfinitely generated projective modules, in contrast, invite little interest; for we show that an obviously necessary "connectedness" condition for such a module to be free is also sufficient.

More precisely, call an R-module P *uniformly* $\aleph$-*big*, where $\aleph$ is an infinite cardinal, if (i) P can be generated by $\aleph$ elements, and (ii) $P/\mathfrak{A}P$ requires $\aleph$ generators for all two-sided ideals $\mathfrak{A}$ ($\neq R$). A free module with a basis of $\aleph$ elements is manifestly uniformly $\aleph$-big. Our main result (Corollary 3.2) asserts, conversely, that, with suitable chain conditions on R, *a uniformly big projective R-module is free.*

Finally, we ask, for what R are all nonfinitely generated projective modules uniformly big? For commutative rings, the answer is quite satisfactory; with mild assumptions one requires only that spec (R) be connected (i.e., that there exist no nontrivial idempotents). For $R = Z\pi$, with π a finite group, Swan (unpublished) has established this conclusion when π is solvable, and it is undoubtedly true in general.

Our method relies on two basic tools. One, naturally enough, is Kaplansky's remarkable theorem [2, Theorem 1] which asserts that every projective module is a direct sum of countably generated modules. The second is an elegant little swindle, observed several years ago by Eilenberg, and which might well have sprung from the brow of Barry Mazur. It is this result, recorded below, which permits us to waive the delicate arithmetic questions which plague the finitely generated case.

EILENBERG'S LEMMA. *If $P \oplus Q = F$ with F a nonfinitely generated free module, then $P \oplus F \cong F$.*

Proof.

$$F \cong F \oplus F \oplus \cdots = P \oplus Q \oplus P \oplus Q \oplus \cdots$$
$$\cong P \oplus F \oplus F \oplus \cdots \cong P \oplus F.$$

Finally, I wish to thank Peter Freyd for several helpful conversations, and, in particular, for pointing out the method in §2.

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