# THE ANALOGUE OF THE PISOT-VIJAYARAGHAVAN NUMBERS IN FIELDS OF FORMAL POWER SERIES[1]

BY

Paul T. Bateman and Alfred L. Duquette

## Introduction

A Pisot-Vijayaraghavan number (PV number) is an algebraic integer $\theta$ greater than 1 all of whose conjugates, other than $\theta$ itself, lie in the open unit disc $|z| < 1$. Vijayaraghavan [9] proved that there exist PV numbers of all degrees, and Pisot [7] proved that in every real algebraic number field there exist PV numbers which generate the field.

The PV numbers have the following basic property, as is easily seen by considering the trace. If $\theta$ is a PV number and $\lambda$ is an algebraic integer in the field generated by $\theta$, then $\| \lambda\theta^n \|$ goes (exponentially) to zero as $n \to +\infty$. Here $\| \lambda\theta^n \|$ denotes the difference, taken positively, between $\lambda\theta^n$ and the nearest integer.

On the other hand, this property of the PV numbers characterizes them to a considerable extent, as the following two results show. Hardy [3] and Vijayaraghavan [9] proved that if $\theta$ is an *algebraic* number greater than 1, if $\lambda$ is a nonzero real number, and if $\| \lambda\theta^n \| \to 0$ as $n \to +\infty$, then $\theta$ is a PV number, and $\lambda$ is in the algebraic number field generated by $\theta$. Secondly, Pisot [6] proved that if $\theta$ is a *real* number greater than 1, if $\lambda$ is a nonzero real number, and if $\sum_{n=1}^{\infty} \| \lambda\theta^n \|^2$ converges, then $\theta$ is algebraic and therefore a PV number. An exposition of these two results may be found in [1, Ch. 8]. A comprehensive bibliography is given in [8].

It is reasonable to conjecture that we can suppress the hypothesis that $\theta$ is algebraic in the first of these two theorems or, equivalently, that we can replace the hypothesis of the convergence of $\sum_{n=1}^{\infty} \| \lambda\theta^n \|^2$ in the second theorem by the assumption that $\lim_{n\to+\infty} \| \lambda\theta^n \| = 0$. In fact this is the principal unsolved problem of the theory at the present time (cf. [10]).

In this paper we construct an analogous theory in the following parallel situation (cf. [4]). In place of the rational integers we consider the ring $k[x]$ of polynomials in an indeterminate $x$ with coefficients in a given field $k$. In place of the field of rational numbers we consider the field $k(x)$ of rational functions in $x$ with coefficients in $k$. In place of the field of real numbers we consider the field $k\{x^{-1}\}$ of pole-like formal Laurent series about $\infty$ with

Received July 21, 1961.

[1] This work was supported by the Office of Naval Research through a contract with the University of Illinois, and also by the Jet Propulsion Laboratory of the California Institute of Technology under a program sponsored by the National Aeronautics and Space Administration.