# CONFORMAL TRANSFORMATIONS OF COMPACT RIEMANNIAN MANIFOLDS

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In his recent paper [3], H. Yamabe has proved that every Riemannian metric on a compact manifold of dimension $\geqq 3$ can be deformed conformally to a Riemannian metric of constant scalar curvature. This makes Riemannian manifolds of constant scalar curvature important in the study of conformal transformations of compact Riemannian manifolds. Indeed, a conformal transformation of a compact Riemannian manifold onto another can be considered as a conformal one between Riemannian manifolds of constant scalar curvature by suitable conformal change of metrics.

On the other hand, K. Yano and T. Nagano [6] have proved that every complete Einstein space of dimension $\geqq 3$ admitting a one-parameter group of global conformal transformations is isometric to an ordinary sphere. Furthermore similar results have been obtained by S. Ishihara and Y. Tashiro [1] in the case of so-called concircular transformations of complete Riemannian manifolds of constant scalar curvature.

The purpose of this paper is to obtain a necessary and sufficient condition for a conformal transformation of a compact Riemannian manifold onto another to be affine (or homothetic) with applications to the case of nonpositive constant scalar curvature. Namely the condition will be obtained in terms of scalar curvatures.

## 1. Preliminaries

Throughout this paper the dimensions of Riemannian manifolds are assumed to be greater than one and every quantity to be class $C^\infty$ as well as the manifolds.

Let $(M, g)$ and $(M', g')$ be $n$-dimensional Riemannian manifolds with Riemannian metrics $g$ and $g'$ respectively. A homeomorphism $f$ of $M$ onto $M'$ is called a *conformal* one of $(M, g)$ onto $(M', g')$ if the Riemannian metric $g^* = f_* g'$ induced from $g'$ by $f$ is conformally related with $g$, i.e., if there exists a scalar function $\phi$ on $M$ such that $g^* = e^{2\phi} g$. If $\phi$ is constant, $f$ is *homothetic*, and if especially $\phi$ is identically zero, $f$ is an *isometry*.

Now if we denote by $K_g$ the scalar curvature of $g$, it is known that the transform $f_* K_{g'}$ of the scalar curvature $K_{g'}$ of $g'$ by $f$ coincides with the scalar curvature of $g^*$, i.e., $f_* K_{g'} = K_{g^*}$. It is also a classical fact that $K_{g^*}$ and $K_g$ are related by the formula

$$(1) \qquad e^{2\phi} K_{g^*} - K_g = 2(n-1)\Delta\phi - (n-1)(n-2)\,|\,d\phi\,|^2,$$

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