DIOPHANTINE PROBLEMS INVOLVING POWERS MODULO ONE[1]

BY
EDWARD C. POSNER

1. Introduction

In [1] the following two results were proved: If θ is a real number and the fractional parts of three different positive integral powers of θ are equal, then θ to some positive integral power is a rational integer; if the fractional parts of two different powers of θ are equal for infinitely many pairs of powers, then the same conclusion follows. Here we give different proofs of these results, but the main purpose of this paper is to generalize the second result to complex numbers, and to apply this generalization to problems in diophantine equations.

2. The real case

We begin with a new proof for the real case. If the fractional part of θ^m equals the fractional part of θ^n, then $\theta^m - \theta^n - a = 0$ for some rational integer a, and conversely. We hereafter assume $m > n$, $\theta > 1$, so that $a > 0$. The case $\theta < -1$ is similar. Observe that an equation $x^m - x^n - a = 0$ with $m > n$, $a > 0$, has exactly one positive root, and this root θ is greater than 1. Also, the set of roots of this equation of largest absolute value is readily seen to be $\{\varepsilon_r \theta\}$, $r = (m, n)$, $\{\varepsilon_r\}$ the set of r r^{th} roots of unity. We now prove Theorems 1 and 2 together.

THEOREM 1. *The fractional parts of three different positive integral powers of a real number θ are equal only when θ is the q^{th} root of a rational integer for some positive integer q.*

THEOREM 2. *The fractional parts of two different positive integral powers of a real number θ are equal for only a finite number of pairs of powers unless θ is the q^{th} root of an integer for some positive integer q.*

Proofs. We assume that we have such a θ not the q^{th} root of a rational integer and arrive at a contradiction. Let θ (> 1) satisfy $f(\theta) = 0$, where $f(x) = x^p + a_{p-1} x^{p-1} + \cdots + a_0$, a_i rational integers, $0 \leqq i \leqq p - 1$, is the monic irreducible equation satisfied by θ over the rationals. (We have integer coefficients because θ is an algebraic integer, satisfying as it does at least one equation $\theta^m - \theta^n - a = 0$, a a rational integer.) Since θ itself is not rational, $p > 1$. Now since f divides the polynomial $x^m - x^n - a$, a a rational integer (because $\theta^m - \theta^n - a = 0$), we conclude by the remarks

Received June 30, 1960; received in revised form April 27, 1961.

[1] A portion of the work reported herein was conducted for the Jet Propulsion Laboratory of the California Institute of Technology under a program sponsored by the National Aeronautics and Space Administration.