RISES AND UPCROSSINGS OF NONNEGATIVE MARTINGALES

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1. Introduction

Let f_n be a stochastic process such as one that describes the successive fortunes of a gambler, the successive prices of a stock, or the population of a particular species. Such processes are *nonnegative*. For each positive real number y, the process experiences a *rise* of size y if for some r and s with $r < s, f_s - f_r \geqq y$. Let x be a positive real number. If $f_0 \equiv x$, the process *begins* at x. In lieu of the semimartingale terminology we sometimes find it suggestive to call a process *subfair* or (*conditional*) *expectation-decreasing* if for all r and s with $r < s$, the conditional expectation of f_s given f_n for $n \leqq r$ does not exceed f_r.

(1.1) THEOREM. *Let f_n, $n = 0, 1, 2, \cdots$, be a nonnegative subfair process that begins at the positive real number x. Then, for each positive real number y, the probability that the process experiences a rise of size y is strictly less than $1 - e^{-x/y}$. Moreover, this bound is best possible.*

The main purpose of this paper is to prove Theorem (1.1), or rather, its generalization, Theorem (11.1), which gives sharp bounds to the probability that nonnegative expectation-decreasing processes experience several rises. Though the first eleven sections of this paper are needed for the proof of (11.1), some of the intermediate results are of interest in themselves. Some of the ideas used in proving the "concrete" results (1.1) and (11.1) have been isolated, and presented in a somewhat general and abstract form in Sections 3, 4, and 6. These same ideas and techniques are then easily applied in Sections 12, 13, and 14, to find sharp bounds to the probability that nonnegative lower semimartingales have k or more upcrossings or downcrossings. These latter sections make contact with earlier work of Doob and Hunt [4], [10].

2. The bound in Theorem (1.1) is best possible

Let f_n be the fortune at time n of a gambler who gambles according to a scheme about to be described. Consider a fair two-valued gamble g that wins $y > 0$ with probability W, and that loses $s > 0$ with probability L. Here $W + L = 1$. Since g is fair,

$$(2.1) \qquad W = \frac{s}{s + y}; \qquad L = \frac{y}{s + y} = \frac{1}{1 + s/y}.$$

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