A DECOMPOSITION THEOREM FOR SUPERMARTINGALES

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It is a well known fact (see [1], p. 296) that any discrete-parameter supermartingale $\{X_n\}_{n \in \mathbf{N}}$ can be represented as a sum:

$$X_n = Y_n + Z_n ,$$

$\{Y_n\}$ being a martingale, and $\{Z_n\}$ a process with decreasing sample functions, such that $Z_0 = 0$. Moreover, if the supermartingale $\{X_n\}$ is uniformly integrable, the same is true for $\{Y_n\}$ and $\{Z_n\}$. Doob has raised the problem of the existence of such a decomposition for continuous-parameter supermartingales. We shall solve this problem here, although the necessary and sufficient condition we give is not very easy to handle. Our proof has been adapted from that of a theorem in potential theory, concerning the representation of excessive functions as potentials of additive functionals ([3], pp. 75–83). A reader with some knowledge of Hunt's potential theory for Markov processes, and the theory of additive functionals, will easily recognize here some kind of a coarse potential theory, with supermartingales replacing excessive functions. Our terminology has been chosen in accordance with this idea.

We shall use freely the results contained in Chapter VII (martingale theory) of Doob's book. A number of definitions will be recalled, for the reader's convenience.

1. Let Ω be a set, $\mathfrak{F}$ a Borel field of subsets of Ω, $\mathbf{P}$ a probability measure defined on $(\Omega, \mathfrak{F})$. We are given a family $\{\mathfrak{F}_t\}_{t \in \mathbf{R}_+}$ of Borel subfields of $\mathfrak{F}$, such that

$$\mathfrak{F}_s \subset \mathfrak{F}_t , \qquad s < t.$$

We may, and do, suppose that the Borel field $\mathfrak{F}$ has been completed with respect to $\mathbf{P}$, and that each $\mathfrak{F}_t$ contains all $\mathfrak{F}$ sets of measure zero. A measurable stochastic process $\{X_t\}_{t \in \mathbf{R}_+}$ is *well adapted* to the $\mathfrak{F}_t$ family if, for each t, X_t is $\mathfrak{F}_t$-measurable. Let $\mathfrak{F}_{t_+}$ denote the intersection $\bigcap_{s>t} \mathfrak{F}_t$; any process which is well adapted to the $\mathfrak{F}_t$ family is well adapted to the $\mathfrak{F}_{t_+}$ family.

A *supermartingale* (relative to the $\mathfrak{F}_t$ family) is a real valued process $\{X_t\}$, well adapted to the $\mathfrak{F}_t$ family, such that

(i) $\forall_t , \quad \mathbf{E}[| X_t |] < \infty,$

(ii) $\forall_s , \quad \forall_t , \quad \mathbf{E}[X_{s+t} \mid \mathfrak{F}_s] \leqq X_s$ a.s.

If equality holds a.s. in (ii), the process is a *martingale*. We shall be concerned here only with sample right continuous supermartingales. If $\{X_t\}$ is such a supermartingale, then (ii) holds with $\mathfrak{F}_t$ replaced by $\mathfrak{F}_{t_+}$. Let indeed A be an event in $\mathfrak{F}_{s_+}$, and let s_n be a decreasing sequence which converges to s.

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