IMAGINARY QUADRATIC FIELDS WITH UNIQUE FACTORIZATION

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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1. Introduction

Nine imaginary quadratic fields are known in which the ring of integers has unique factorization, namely the fields with discriminants

$$-4,\ -8,\ -3,\ -7,\ -11,\ -19,\ -43,\ -67,\ -163.$$

Heilbronn and Linfoot [3] proved that there can exist at most one more such field. Dickson [2] showed that if this tenth field actually exists, then its discriminant must be numerically greater than 1 500 000, while Lehmer [5] improved this bound to 5 000 000 000.

It is easy to prove (see the last footnote on p. 294 of [3]) that if an imaginary quadratic field other than those with discriminants -4 and -8 has unique factorization, then its discriminant must be of the form $-p$, where p is a prime congruent to 3 modulo 4. We shall use $h(-p)$ to denote the number of classes of ideals in the ring of integers of the imaginary quadratic field with discriminant $-p$, and $L_p(s)$ to denote the Dirichlet L-function formed from the unique real nonprincipal residue-character modulo p. The latter is given by the formulas

$$L_p(s) = \sum_{n=1}^{\infty} \left(\frac{-p}{n}\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right)\frac{1}{n^s} \qquad (s > 0)$$

in terms of the Kronecker and Legendre symbols respectively.

There are various results showing that if $h(-p) = 1$ for some prime p greater than 163, then $L_p(s)$ must have a real zero rather close to 1. For example, S. Chowla and A. Selberg [1] showed that if $h(-p) = 1$ for some prime p greater than 163, then $L_p(\frac{1}{2}) < 0$ and so $L_p(s)$ has a real zero between $\frac{1}{2}$ and 1 (since $L_p(1)$ is positive).

A more specific result follows from an inequality of Hecke, which is proved in [4]. If $0 < a \leqq 2$ and $L_p(s)$ has no real zeros greater than $1 - a/\log p$, Hecke showed that

$$h(-p) > \frac{a}{11000}\frac{p^{1/2}}{\log p}.$$

(This is trivial if $p < 10^{10}$, and otherwise follows from the inequality at the

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