ON THE NUMBER OF INTEGERS $\leqq x$ WHOSE PRIME FACTORS DIVIDE n

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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If n and x are positive integers, then we let $f(n, x)$ denote the number mentioned in the title, i.e., the number of integers m with $1 \leqq m \leqq x$, $m \mid n^\infty$. (The notation $m \mid n^\infty$ means that m divides some power of n, or in other words, that all prime factors of m divide n.)

P. Erdös conjectured (in a letter to the author, December 2, 1960) that the average $M(x) = x^{-1}\sum_{n=1}^{x} f(n, x)$ can be written as

$$M(x) = x^{-1}F(x) = \exp((\log x)^{1/2+\varepsilon(x)}), \qquad \text{where } \varepsilon(x) \to 0 \text{ for } x \to \infty.$$

We shall show in this paper (Theorem 2) that this is true. In fact we can get a much more precise result, viz. that $\log M(x)$ is asymptotically equivalent to $(8 \log x)^{1/2}(\log \log x)^{-1/2}$. Needless to say, this is still very far from an asymptotic formula for $M(x)$ itself.

The asymptotic formula for the logarithm of the average does not change if we replace $\sum_{n=1}^{x} f(n, x)$ by $\sum_{n=1}^{x} f(n, n)$, which is also considered in Theorem 2. This may give an idea of how rough our result still is.

We shall derive Theorem 2 from Theorem 1, which has some interest in itself. It deals with the partial sums of the series that results from the harmonic series if every denominator n is replaced by the product of the primes that divide it. This result will be obtained in a classical way: We build the corresponding Dirichlet series $f(\sigma)$ (see Lemma 2), we derive asymptotic information about $f(\sigma)$ if $\sigma \to 0$ (provided by Lemma 1), and we translate this information into information concerning the partial sums. This translation is achieved by a Tauberian theorem of Hardy and Ramanujan (see Lemma 3).

LEMMA 1. *Let h be a positive constant. If $\sigma > 0$, we define*

$$A_h(\sigma) = \int_{3/2}^{\infty} \{\log (1 + x^{-1}(x^\sigma - 1)^{-1})\}(\log x)^{-h}\, dx.$$

Then we have, if $\sigma \to 0$, $\sigma > 0$, and if h is fixed,

$$A_h(\sigma) = h^{-1}\sigma^{-1}(\log \sigma^{-1})^{-h} + O\{\sigma^{-1}(\log \sigma^{-1})^{-h-1} \log \log \sigma^{-1}\}.$$

Proof. Throughout this proof we abbreviate

$$(\log \sigma^{-1})^{-1} = \eta.$$

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