ON A THEOREM OF RADEMACHER-TURÁN

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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A set of points some of which are connected by an edge will be called a graph G. Two vertices are connected by at most one edge, and loops (i.e., edges whose endpoints coincide) will be excluded. Vertices will be denoted by $\alpha, \beta, \cdots$, edges will be denoted by $e_1, e_2, \cdots$ or by (α, β) where the edge (α, β) connects the vertices α and β.

$G - e_1 - \cdots - e_k$ will denote the graph from which the edges $e_1, \cdots, e_k$ have been omitted, and $G - \alpha_1 - \cdots - \alpha_k$ denotes the graph from which the vertices $\alpha_1, \cdots, \alpha_k$ and all the edges emanating from them have been omitted; similarly $G + e_1 + \cdots + e_k$ will denote the graph to which the edges $e_1, \cdots, e_k$ have been added (without generating a new vertex).

The valency $v(\alpha)$ of a vertex will denote the number of edges emanating from it. $G_u^{(v)}$ will denote a graph having v vertices and u edges. The graph $G_{\binom{k}{2}}^{(k)}$ (i.e., the graph of k vertices any two of which are connected by an edge) will be called the complete k-gon.

A graph is called *even* if every circuit of it has an even number of edges.

Turán[1] proved that every

$$G_{V+1}^{(n)}, \qquad V = \frac{k-2}{2(k-1)}(n^2 - r^2) + \binom{r}{2}$$

for $n = (k-1)t + r, 0 \leqq r < k - 1$, contains a complete k-gon, and he determined the structure of the $G_V^{(n)}$'s which do not contain a complete k-gon. Thus if we put $f(2m) = m^2, f(2m + 1) = m(m + 1)$, a special case of Turán's theorem states that every $G_{f(n)+1}^{(n)}$ contains a triangle.

In 1941 Rademacher proved that for even n every $G_{f(n)+1}^{(n)}$ contains at least $[n/2]$ triangles and that $[n/2]$ is best possible. Rademacher's proof was not published. Later on[2] I simplified Rademacher's proof and proved more generally that for $t \leqq 3, n > 2t$, every $G_{f(n)+t}^{(n)}$ contains at least $t[n/2]$ triangles. Further I conjectured that for $t < [n/2]$ every $G_{f(n)+t}^{(n)}$ contains at least $t[n/2]$ triangles. It is easy to see that for $n = 2m, 2m > 4$, the conjecture is false for $t = n/2$. To see this, consider a graph $G_{m^2+m}^{(2m)}$ whose vertices are

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[1] P. Turán, Matematikai és Fizikai Lapok, vol. 48 (1941), pp. 436–452 (in Hungarian); see also *On the theory of graphs*, Colloq. Math., vol. 3 (1954), pp. 19–30.

[2] P. Erdös, *Some theorems on graphs*, Riveon Lematematika, vol. 9 (1955), pp. 13–17 (in Hebrew with English summary).