APPROXIMATE FORMULAS FOR SOME FUNCTIONS OF PRIME NUMBERS

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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2. Introduction

Counting 2 as the first prime, we denote by $\pi(x)$, $\vartheta(x)$, and $\psi(x)$, respectively, the number of primes $\leqq x$, the logarithm of the product of all primes $\leqq x$, and the logarithm of the least common multiple of all positive integers $\leqq x$; if $x < 2$, we take $\pi(x) = \vartheta(x) = \psi(x) = 0$. We also let p_n denote the n^{th} prime, and $\phi(n)$ denote the number of positive integers $\leqq n$ and relatively prime to n. Throughout, n shall denote a positive integer, p a prime, and x a real number. We shall present approximate formulas for $\pi(x)$, $\vartheta(x)$, $\psi(x)$, p_n, $\phi(n)$, and other functions related to prime numbers.

In 1808, on the basis of attempting to fit known values of $\pi(x)$ by an empirical formula, Legendre conjectured an approximation very similar to that given below in (2.19). In 1849, again on the basis of counts of the number of primes in various intervals, Gauss communicated to Encke a conjecture that in the neighborhood of the number x the average density of the primes is $1/\log x$. On this basis, if one should wish an estimate for the sum of $f(p)$ over all primes $p \leqq x$, the natural approximation would be

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