CONGRUENCES FOR THE PARTITION FUNCTION TO COMPOSITE MODULI[1]

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Let $p(n)$ denote the number of unrestricted partitions of the integer n, so that

$$\sum_{n=0}^{\infty} p(n)x^n = \phi(x)^{-1}, \qquad \phi(x) = \prod_{n=1}^{\infty} (1 - x^n).$$

In a recent article [3] the conjecture was made that for all integers $m \geqq 2$ each of the m congruences

$$p(n) \equiv r \pmod{m}, \qquad 0 \leqq r \leqq m - 1,$$

has infinitely many solutions in positive integers n; and a proof of this conjecture was given for $m = 2, 5, 13$. The principal object of this note is to prove that the conjecture holds for $m = 65$ as well. Certain related congruences will also be proved.

The writer's interest in these matters (which have their origin in the famous Ramanujan congruences) was first awakened by H. Rademacher, and this note is dedicated to him.

It is convenient to introduce some notation and to reproduce some known material here. If n is a nonnegative integer, define $p_r(n)$ as the coefficient of x^n in $\phi(x)^r$; otherwise define $p_r(n)$ as 0. Thus $p(n) = p_{-1}(n)$. Then it is known (see [4], [7]) that

$$p(13n + 6) \equiv 11p_{11}(n) \pmod{13}. \tag{1}$$

The author has shown in [5] that if r is odd, $1 \leqq r \leqq 23$, and p is a prime such that

$$r\nu = r(p^2 - 1)/24 \quad \text{is an integer,}$$

then for all integral n

$$p_r(np^2 + r\nu) - \gamma_n p_r(n) + p^{r-2}p_r((n - r\nu)/p^2) = 0, \tag{2}$$

where

$$\gamma_n = c - \chi(r\nu - n)p^{(r-3)/2}(-1)^{(p-1)(p-1-2r)/8},$$
$$c = p_r(r\nu) + \chi(r\nu)p^{(r-3)/2}(-1)^{(p-1)(p-1-2r)/8},$$

and χ is the Legendre-Jacobi quadratic reciprocity symbol modulo p.

It is easy to deduce from (2) that if

$$a_n = r(p^{2n} - 1)/24, \qquad t_n = p_r(a_n),$$

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