SYMMETRY TYPES OF PERIODIC SEQUENCES

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1. Introduction

This paper gives a short treatment of the problem appearing in Fine [2], which is as follows. Consider periodic sequences \( a = (\cdots, a_{-1}, a_0, a_1, \cdots) \) with period \( n \) and with \( a_j \) limited to the \( q \) values 1, 2, \( \cdots, q \). If two sequences are taken to be equivalent when they can be made alike either by a shift in origin or by a permutation of the element values 1, 2, \( \cdots, q \), or by both, how many distinct (inequivalent) sequences, or symmetry types of sequences are there?

An example given by Fine is repeated here for concreteness. For \( n = 3 \), \( q = 2 \) there are two types, namely (111) and (112); (111) and (222) are equivalent by the permutation \((12)\), and the six remaining sequences (112), (121), (211), (221), (212), (122), are equivalent either by this permutation or a shift in origin.

Section 4 is devoted specifically to Fine’s problem. Depending on the intended application, a group \( G \) of symmetry transformations (possibly different from Fine’s) may be allowed. If only translations \((a_i \to a_{i+1})\) are allowed, \( G \) is a cyclic group \( C_n \). This case appears in [5] in connection with counting necklaces made from \( n \) beads of \( q \) different kinds (translations merely rotate the necklace). It also arises in problems of coding and genetics [3]. The special case \( n = 12 \), \( q = 2 \) occurs in finding the number of distinct musical chords (of 0, 1, \( \cdots \), or 12 notes) when inversions and transpositions to other keys are equivalences. Turning over the plane of necklace \((a_i \to a_{-i})\) produces a new “mirror image” necklace. If this symmetry is permitted as well as the translations, then \( G \) is a dihedral group \( D_n \). Permutations of the element values 1, 2, \( \cdots, q \) form a symmetric group \( S_q \). Thus, in Fine’s problem, \( G \) is a product group \( C_n \times S_q \). This problem has some applications to switching theory. For example, consider a switching network to control \( q \) lights, one at a time, in a periodic cycle; here \( a_i \) is the name of the light which changes its state at the \( i^{th} \) step. In counting the number of distinct sequences possible, translations merely start the cycle at a different point and permutations of 1, \( \cdots, q \) merely give the lights new names. If sequences which operate the lights in reverse order are also considered equivalent, then \( G \) becomes \( D_n \times S_q \). More details on the music and switching applications appear in Section 6.

Our treatment of \( C_n \times S_q \) is related to a special case of one of the theorems in de Bruijn [1]. By its use it is also easy to treat the case \( D_n \times S_q \).

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