SYMMETRY TYPES OF PERIODIC SEQUENCES

BY

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1. Introduction

This paper gives a short treatment of the problem appearing in Fine [2], which is as follows. Consider periodic sequences $a = (\cdots, a_{-1}, a_0, a_1, \cdots)$ with period n and with a_j limited to the q values $1, 2, \cdots, q$. If two sequences are taken to be equivalent when they can be made alike either by a shift in origin or by a permutation of the element values $1, 2, \cdots, q$, or by both, how many distinct (inequivalent) sequences, or symmetry types of sequences are there?

An example given by Fine is repeated here for concreteness. For $n = 3$, $q = 2$ there are two types, namely (111) and (112); (111) and (222) are equivalent by the permutation (12), and the six remaining sequences (112), (121), (211), (221), (212), (122), are equivalent either by this permutation or a shift in origin.

Section 4 is devoted specifically to Fine's problem. Depending on the intended application, a group G of symmetry transformations (possibly different from Fine's) may be allowed. If only translations ($a_i \rightarrow a_{i+s}$) are allowed, G is a cyclic group C_n. This case appears in [5] in connection with counting necklaces made from n beads of q different kinds (translations merely rotate the necklace). It also arises in problems of coding and genetics [3]. The special case $n = 12$, $q = 2$ occurs in finding the number of distinct musical chords (of 0, 1, $\cdots$, or 12 notes) when inversions and transpositions to other keys are equivalences. Turning over the plane of necklace ($a_i \rightarrow a_{-i}$) produces a new "mirror image" necklace. If this symmetry is permitted as well as the translations, then G is a dihedral group D_n. Permutations of the element values $1, 2, \cdots, q$ form a symmetric group S_q. Thus, in Fine's problem, G is a product group $C_n \times S_q$. This problem has some applications to switching theory. For example, consider a switching network to control q lights, one at a time, in a periodic cycle; here a_i is the name of the light which changes its state at the i^{th} step. In counting the number of distinct sequences possible, translations merely start the cycle at a different point and permutations of $1, \cdots, q$ merely give the lights new names. If sequences which operate the lights in reverse order are also considered equivalent, then G becomes $D_n \times S_q$. More details on the music and switching applications appear in Section 6.

Our treatment of $C_n \times S_q$ is related to a special case of one of the theorems in de Bruijn [1]. By its use it is also easy to treat the case $D_n \times S_q$.

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